QUANTIZATION IN SPACES OF CONSTANT CURVATURE

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A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

1978

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A Thesis Presented by

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To the

University of St Andrews

In application for the degree of

Doctor of Philosophy

January 1978

Declaration

The accompanying thesis is my own composition. It is based on work carried out by me and no part of it has previously been presented in application for a higher degree.

Certificate

I certify that the conditions of the Ordinance and Regulations have been fulfilled.

Research Supervisor

ABSTRACT

Quantization in generalized coordinate systems and in non-Euclidean spaces has attracted much recent attention. The aim of this thesis is to discuss the problem for spaces of constant curvature.

Chapter I is a brief review of tensor analysis on manifolds and the Hamiltonian formulation of classical mechanics.

Chapter II deals with the canonical quantization scheme. It is proved that this scheme is unsatisfactory since it often fails to produce an essentially self-adjoint operator corresponding to a generalized momentum. It is shown that the only case for which we can obtain a unique well-defined pair of canonical quantum observables is that when the range of the generalized coordinate is the entire real line.

Chapter III is devoted for discussing the quantization in spaces of constant curvature. An approach by Mackey circumvents the difficulties facing the canonical quantization scheme and is adopted instead. It is shown that a more physical meaning is gained if we impose on Mackey's momenta the condition that they are metric-preserving. Such a condition makes the momenta constants of the free motion. §4 in essence demonstrates that the dynamics in a space of constant curvature are rooted in its geometry. Utilizing the metric-preserving momenta, the quantum Hamiltonian is built up from symmetry considerations, and is uniquely determined as the Laplacian up to a multiplicative constant and an additive constant. In §5-§8 the quantization in spaces of constant curvature is studied in detail. The quantum and classical momenta are found explicitly. Eigenvalues and eigenfunctions of the momentum observables are evaluated. Also it is shown that spaces of different curvature are physically distinguishable. An interesting relation between the momenta, the

curvature of the space and the Hamiltonian holds classically and quantum-mechanically. The well-known relation giving the Hamiltonian as proportional to the sum of squares of the momenta (in Euclidean spaces with Cartesian coordinates) is just a special case of a more general relation in which the momenta, the angular momenta and the curvature of the space take part in forming the Hamiltonian classically or quantum-mechanically. §9 of chapter III discusses the Lie algebraic treatment of quantization. It is shown that such a treatment cannot be taken as a general quantization scheme. The sets of all classical and quantal momenta (metric-preserving and non-metric-preserving) do not form Lie algebras. However when we confine ourselves to those momenta which are generated by motions of the space, then these sets form isomorphic Lie algebras. The striking parallelism between quantum and classical mechanics observed throughout §4-§8 is pushed further in §9. It is shown that the quantum and classical Hamiltonians are, respectively, related to the Casimir operators of the Lie algebras of quantum and classical momenta in the same way.

Chapter IV envisages the problem of quantization from an intrinsic point of view. A space of constant curvature is looked on as a hypersurface in a flat space. In §4 we verify that Dirac's scheme for quantizing a constrained classical system works when the constraints are geometrical.

Acknowledgements

I wish to express my sincere gratitude to my supervisor Dr Kay-Kong Wan. His patient guidance, encouragement and help throughout my work are deeply appreciated.

My gratitude is also due to Professor R.B. Dingle for reading my thesis and commenting on it and for the facilities made available to me in the Department of Theoretical Physics.

Thanks to Dr J.F. Cornwell for acting as my supervisor during Dr Wan's sabbatical leave 1977-1978 and for the useful consultations.

Thanks to Professor E. Copson for the constructive consultation he offered me, and to Dr E. Robertson and Mr Keith McFarlane for the helpful discussions.

Thanks to Miss L. McLean and Miss P. Russell for typing my thesis, and to Mrs I. Bukowska for her kind and efficient help in the use of the library's facilities.

Finally, I wish to thank the Ministry of Higher Education in Syria for providing me with the financial support throughout my work.

To

My Fiancé

Mae Subbagh

Contents

INTRODUCTION

Quantization in generalized coordinate systems and in non-Euclidean spaces has attracted much recent attention. The aim of this thesis is to discuss the problem for spaces of constant curvature.

Chapter I reviews some of the background underlying this work. This review comprises manifold theory, tensor analysis on manifolds and the Hamiltonian formulation of classical mechanics. Considerable attention is given to concepts appearing quite frequently in the course of development of this thesis.

Chapter II deals with the most commonly used scheme for tackling the problem of quantization in generalized coordinates and in curved spaces, that is the canonical quantization scheme. This scheme is proved to be unsatisfactory since it often fails to produce an essentially self-adjoint operator corresponding to a generalized momentum.

Chapter III is devoted to discussing the quantization in spaces of constant curvature. An approach by Mackey circumvents the difficulties facing the canonical quantization scheme and is adopted instead. Mackey's scheme is discussed in §2 and illustrated by examples in §3. It is shown that a more physical meaning is gained if we impose on Mackey's momenta the condition that they are metric preserving, i.e., generated by motions of the space under consideration. Such a condition makes the momenta constants of the free motion.

§4 in essence demonstrates that the dynamics in a space of constant curvature are rooted in its geometry. Utilizing the metric-preserving momenta, the quantum Hamiltonian is built up from symmetry considerations, and is uniquely determined as the Laplacian up to a multiplicative constant and an additive constant. In §5 - §8 the quantization in spaces of constant curvature is studied in detail. The quantum and classical momenta are found explicitly. Eigenfunctions and eigenvalues of the momentum observables are evaluated. Also it is shown that spaces of different curvature are physically distinguishable in the sense that energy measurements can distinguish spaces of different curvatures. An interesting relation between the momenta, the curvature of the space and the Hamiltonian holds classically and quantum-mechanically. The well-known relation giving the Hamiltonian as proportional to the sum of squares of the momenta (in Euclidean spaces with Cartesian coordinates) is just a special case of a more general relation in which the momenta, the angular momenta and the curvature of the space take part in forming the Hamiltonian classically or quantum-mechanically.

Our study of the quantization in spaces of constant curvature starts with the 2-dimensional case in §5, passes by the 3-dimensional case in §6 and finishes with the N-dimensional case in §8. In section 7 we consider the hyperbolic plane and the hyperbolic space as applications to §5 and §6.

Most of the language used in §4 - §8 is geometrical. However, in §9 we will discuss the Lie algebraic treatment of quantization. It will be shown that such a treatment cannot be taken as a general quantization scheme. The sets of all classical and quantal momenta (metric-preserving and non-metric-preserving) do not form Lie algebras. However when we confine ourselves to those momenta which are generated by motions of the space, then these momenta form isomorphic Lie algebras. The striking parallelism between quantum and classical mechanics observed throughout §4 - §8 is pushed further in §9. It is shown that the quantum and classical Hamiltonians are, respectively, related to the Casimir operators of the Lie algebras of quantum and classical momenta in the same way.

Chapter IV envisages the problem of quantization from an extrinsic point of view. A space of constant curvature is looked on as a hypersurface in a flat space. Explicit forms of embedding spaces of constant curvature in higher dimensional flat spaces are obtained in various systems of coordinates used in §5 - §8 of the previous chapter. In §4 we shall consider Dirac's scheme for quantizing a constrained classical system and verify that this scheme works when the constraints are geometrical.

A paper based on Chapter II and §4 - §7 of Chapter III has been published in 'Progress of Theoretical Physics Volume 58, Number 3, pp1030-1044'. A second paper based on §8 and §9 of Chapter III is being submitted to 'Canadian Journal of Physics'.

CHAPTER I

Review of Relevant Material

The aim of this review is to familiarize ourselves with some of the background underlying the work of this thesis. Many concepts are discussed briefly and theorems are quoted without proofs. Full details may be found in references [4, 27-33].

§ 1 Manifolds

§ (1.1) Definition

An N-dimensional topological (or C^0) manifold M is a Hausdorff space with a countable base which is locally homeomorphic to an open set in the N-dimensional space $\mathbb{R}^N$. The expression "locally homeomorphic" means that every point $m \in M$ has an open neighbourhood which is topologically equivalent to an open set of $\mathbb{R}^N$.

Let $m \in M$, then there exists an open neighbourhood $O \ni m$, an open set $E \subseteq \mathbb{R}^N$ and a homeomorphic map $\psi: O \rightarrow E$. The pair (O,ψ) is called a local coordinate system or a chart. For any $m \in O$, we have $\psi(m) = (x^1(m),\ldots, x^N(m))$. The set of real-valued functions x^i defined on O are called the local coordinates of the point $m \in O$ with respect to the chart (O,ψ). The chart (O,ψ) may be denoted by $(x^1(m),\ldots, x^N(m))$, $m \in O$, or simply by $(x^1,\ldots, x^N)$. We note that M is coverable by a global chart if it is homeomorphic to an open set of $\mathbb{R}^N$.

An atlas on M is a collection of charts (O_α,ψ_α) satisfying

(i) $\cup_\alpha O_\alpha = M$,

(ii) $\psi_\alpha \circ \psi_\beta^{-1}$ is C^∞ $\forall \alpha,\beta$.

A chart (O,ψ) is said to be admissible to the atlas $\{(O_\alpha,\psi_\alpha)\}$ if $\psi \circ \psi_\alpha^{-1}$ are C^∞ for all α. A differentiable structure F on M is an atlas on M added to it all its admissible charts.

A C^∞ differentiable manifold (M,F) is a topological manifold together with a differentiable structure F. In future, we will use the word manifold to mean a C^∞ manifold, and will denote it by M. In practice, M is specified by singling out an atlas on M.

We note that we could have defined the so-called C^k manifolds by relaxing the requirement (ii) to become $\psi_\beta \circ \psi_\alpha^{-1}$ is a C^k function in $\mathcal{R}^N$. Unless it is indicated to the contrary, the term differentiability when it refers to M or to functions on M will mean C^∞ differentiability, though many discussions do not necessitate such a stringent condition on M or on functions on it.

§ (1.2) Differentiable Functions and Differential Maps

A function $f:M \to \mathcal{R}$ is said to be differentiable on M if $f \circ \psi^{-1}$ is differentiable for every chart $(O,\psi) \in F$. The differentiable function $f:M \to \mathcal{R}$ induces a numerical function $\overline{f} = f \circ \psi^{-1}:\psi(O) \subseteq \mathcal{R}^N \to \mathcal{R}$, such that, if $(O,\psi) = (x^1,\ldots,x^N)$, then $\overline{f}$ is differentiable with respect to $x^1,\ldots, x^N$. We will use the symbol f for $\overline{f}$ as well in future. In practice, it is sufficient to require that f is differentiable for every chart in an atlas.

Let M and M' be C^∞ manifolds, π a map from M to M' and f a function defined on M'. Define a function π^*f on M by $\pi^*f(m) = f \circ \pi(m)\ \forall\, m \in M$. Such a function is called the pull-back of f from M' to M.

A map $\pi:M \to M'$ is said to be C^∞ on M if for every C^∞ function f on M', π^*f is also a C^∞ function on M. A diffeomorphism from M to M' is a C^∞ map $\pi: M \to M'$ such that it is 1-1 onto and the inverse map π^{-1}: $M' \to M$ is also C^∞.

§ (1.3) Tangent Vectors, Tangent Spaces, Tangent Bundle and Vector Fields

Let $C^{\infty}(O_m)$ be the set of all C^{∞} real-valued functions defined on some neighbourhood O_m of $m \in M$. A tangent vector v at m is a map from $C^{\infty}(O_m)$ into $\mathbb{R}$ satisfying

$$
\begin{aligned}
&(i)\ v(\alpha f+\beta g) = \alpha v(f) + \beta v(g), \\
&(ii)\ v(fg) = v(f)\, g(m) + f(m)\, v(g),
\end{aligned}
\tag{1.1.1}
$$

$(\forall \alpha, \beta \in \mathbb{R};\ \forall f, g \in C^{\infty}(O_m))$. Define $v_1 + v_2$ and αv by

$$
\begin{aligned}
(v_1 + v_2)(f) &= v_1(f) + v_2(f), \\
(\alpha v)(f) &= \alpha(v(f)),
\end{aligned}
\tag{1.1.2}
$$

where v_1, v_2 and v are tangent vectors at m, $f \in C^{\infty}(O_m)$, and $\alpha \in \mathbb{R}$, then $v_1 + v_2$ and αv are tangent vectors at m. It can easily be verified that the set of tangent vectors at m form a vector space over $\mathbb{R}$, with respect to the operations (1.1.2). The vector space T_m constructed from all tangent vectors at m is called the tangent space at m.

Theorem: if the manifold M is N-dimensional, then so is T_m. If $(x^1, \ldots, x^N)$ is a local chart about m, then the set $\{(\partial/\partial x^1)_m, \ldots, (\partial/\partial x^N)_m\}$ of tangent vectors at m form a basis of T_m [27,28].

In view of this theorem, every tangent vector v_m could be written as

$$v_m = v^i(m)\,(\partial/\partial x^i)_m . \tag{1,1,3}$$

Thus given a local coordinate system on an open set $O \subset M$, a basis of the tangent space at every $m \in O$ is determined. A vector $v \in T_m$ is specified by its N components $(v^1, \ldots, v^N)$ with respect to this basis. The components of v transform according to the law $v'^i = v^j \dfrac{\partial x'^i}{\partial x^j}(m)$, where $(x'^1, \ldots, x'^N)$ is another chart about m. It follows that a tangent vector is a contravariant vector.

Let TM denote the set of all tangent vectors on M, i.e., $TM = \cup_{m\epsilon M} T_m$. An element of TM is specified by the pair (m,v), where $m \epsilon M$ and $v \epsilon T_m$. Thus, every element of TM is specified by 2N numbers $(x^1,\ldots, x^N, v^1,\ldots, v^N)$. TM can be endowed by a manifold structure [4,27]. Endowed with such a structure it is called the tangent bundle.

Define a map $L: M \to TM$ by $L(m) = v_m$, where v_m is some tangent vector at m. Such a map is called a vector field or tangent or contravariant field. The vector field may be expressed in the vicinity of every $m \epsilon M$ as

$$L = \xi^i (x^1,\ldots, x^N) \, (\partial/\partial x^i), \tag{1.1.4}$$

where $\xi^1,\ldots, \xi^N$ are functions on a neighbourhood of m, and $(x^1,\ldots, x^N)$ is a chart on that neighbourhood. Denote the set of all C^∞ functions defined everywhere on M by $C^\infty(M)$. If $L(f) \epsilon C^\infty(M)$ for every $f \epsilon C^\infty(M)$, then L is called a C^∞ vector field on M. It is clear that L is C^∞ if $\xi^i \epsilon C^\infty(M)$ for all i.

The set of all C^∞ functions on M form an associative algebra under the operations of multiplication by a scalar, summation and multiplication of two elements of it, when these operations are defined in the usual way [28]. We may look on the vector field L as a derivation of the associative algebra $C^\infty(M)$ in the sense that it is a mapping from $C^\infty(M)$ into $C^\infty(M)$ such that

$$\begin{aligned} L(\alpha f + \beta g) &= \alpha L(f) + \beta L(g), \\ L(fg) &= L(f)g + f L(g). \end{aligned} \tag{1.1.5}$$

Conversely, if D is a derivation of $C^\infty(M)$, then there is a unique C^∞ vector field L on M such that D = L. This vector field is obtained by setting $L(m)(f) = D(f)(m)$ [4,28].

§ (1.4) Cotangent Spaces, Cotangent Vectors, Differential of a Function, Cotangent Bundle and Cotangent Vector Fields

The cotangent space T^*_m of a manifold M at a point m is the dual space of the tangent space T_m at m. Every element $w_m \in T^*_m$ is called a cotangent vector of M at m.

Let $f \in C^\infty(O_m)$. The differential $(df)_m$ of f at m is a map $(df)_m : T_m \to \mathbb{R}$ such that

$$(df)_m(v_m) = v_m(f). \tag{1.1.6}$$

$(df)_m$ is a linear functional on T_m, since $(df)_m(\alpha v_1 + \beta v_2) = (\alpha v_1)(f) + (\beta v_2)(f) = \alpha(df)_m(v_1) + \beta(df)_m(v_2)$ for all $v_1, v_2 \in T_m$. Thus it belongs to T^*_m. In particular, if $(x^1,\ldots,x^N)$ is some chart on O_m, then $x^i \in C^\infty(O_m)$, and hence, $(dx^i)_m(\partial/\partial x^j)_m = \delta^i_j$. Thus, the set $\{(dx^1)_m,\ldots, (dx^N)_m\} \subset T^*_m$ is a basis of the cotangent space T^*_m dual to the basis $\{(\partial/\partial x^1)_m,\ldots, (\partial/\partial x^N)_m\}$ of the tangent space T_m. Since $(df)_m \in T^*_m$, it may be expressed as $w_i(m)\,(dx^i)_m$. Making use of such an expression and of (1.1.6), we find

$(\partial f/\partial x^i)(m) = (df)_m(\partial/\partial x^i)_m = (w_j(m))(dx^j)_m)(\partial/\partial x^i)_m =$
$w_j(m)\delta^j_i = w_i(m)$.
Thus $(df)_m = (\partial f/\partial x^i)(m)(dx^i)_m$.

Let $T^*M = \cup_{m\in M} T^*_m$. Every element of T^*M could be specified by a pair (m,w), where $m \in M$ and $w \in T^*_m$, i.e., it is specified by 2N numbers $(x^1,\ldots, x^N, w_1,\ldots, w_N)$. T^*M could be endowed with a manifold structure. The result is a manifold called the cotangent bundle.

A cotangent vector field W on M is a map $W: M \to T^*M$, defined by $W(m) = w_m$. A cotangent vector field W is said to be C^∞ if $W(L) \in C^\infty(M)$ for every C^∞ vector field L on M.

We note that if $f \in C^{\infty}(M)$, then df is defined throughout M and it is a C^{∞} cotangent vector field on M, since $(df)(L) = L(f)$ is a C^{∞} function on M for every C^{∞} vector field L on M. A general C^{∞} cotangent vector field W on M could be expressed locally as $w_i(x^1,\ldots,x^N)\,dx^i$, where $w_i (i = 1,\ldots,N)$ belong to $C^{\infty}(M)$ and $(x^1,\ldots,x^N)$ is some chart on M. However, not every cotangent vector field could be expressed as a differential function on M. Those which can be put in the form df where f is some function on M are called exact.

The components $w_i(x^1,\ldots,x^N)$ of W transform according to the law

$$\omega'_i = \omega_j \frac{\partial x^j}{\partial x'^i} . \tag{1.1.7}$$

Hence, cotangent vector fields may be referred to as covariant vector fields.

§ (1.5) Curves in M, integral Curves of a Vector Field, and One-Parameter Groups of Transformations.

A differential curve in M is a C^{∞} map σ from an open interval $(a,b) \subseteq \mathbb{R}$ into M. If $\sigma(t) = m$, and $(x^1,\ldots,x^N)$ is some chart about m, then the pull-backs of $x^i (i = 1,\ldots,N)$ to (a,b), namely $\sigma^* x^i$, are C^{∞} functions on an open subinterval $I \subseteq (a,b)$. The vector $(d\sigma^i/dt)(m)(\partial/\partial x^i)_m$, where $\sigma^i = \sigma^* x^i$, satisfies all the conditions laid down in defining a tangent vector at m. We call such a vector a tangent vector to the curve σ at m.

Let $L = \xi^i \partial/\partial x^i$ be a vector field on M and σ be a differentiable curve in M defined on (a,b). Then σ is said to be an integral curve of L if $(d\sigma^i/dt)(\partial/\partial x^i)_{\sigma(t)} = L(\sigma(t))$ for all $t \in (a,b)$. Equivalently, the differential equations

$$\frac{d\sigma^i}{dt} = \xi^i(\sigma^1,\ldots,\sigma^N) \quad (i=1,\ldots,N),$$

must be satisfied. We note that a reparametrization $\phi(a,b) \to (\alpha,\beta)$ of an integral curve is also an integral curve iff it is a translation of the parameter t [27].

If L is a C^∞ vector field (actually C^1 is sufficient), then it determines a unique curve through each $m \in M$ as the following theorem states:

Theorem:if σ_1 and σ_2 are integral curves of L defined on I_1 and I_2 respectively, and if $\sigma_1(0) = \sigma_2(0)$, then $\sigma_1 = \sigma_2$ for each point in $I_1 \cap I_2$ [28].

Let U_t be a diffeomorphic transformation of M (or simply transformation for short) defined for each $t \in \mathcal{R}$ and satisfying

(i) $U_t \circ U_s = U_{t+s}$ ($t, s \in \mathcal{R}$),

(ii) $(t,m) \to U_t(m)$ is a differentiable map from $\mathcal{R} \times M$ onto M.

The family $\{U_t\}$ is called a one-parameter group of transformations of M (OPG for short). In fact $\{U_t\}$ form an abelian group under the composition law defined by (i). The identity element of $\{U_t\}$ is U_o and the inverse element of U_t is $U_t^{-1} = U_{-t}$. When there is no ambiguity $\{U_t\}$ will be denoted by U_t or just by U.

The set of points $\{U_t(m) | t \in \mathcal{R}\}$ is called the orbit of the OPG through m. Hence, for a fixed m, the curve $\sigma_m(t) = U_t(m)$ is differentiable by (ii) and the image of σ_m is the orbit of m.

An OPG of M defines a C^∞ vector field L whose value at m is given by

$$L(m) = d[U_t^i(x^1, \ldots, x^N)]/dt\,\big|_{t=0}\ \partial/\partial x^i\ , \ (U_t^i = x^i \circ U_t). \qquad (1,1,8)$$

The vector field is called the infinitesimal generator of U_t. It is clear that $\sigma_m(t) = U_t(m)$ is an integral curve of L with $\sigma(0) = m$.

A vector field L is said to be complete if its integral curves through every $m \in M$ are defined on the entire real line $\mathbb{R}$. The infinitesimal generator of an OPG U_t of M is complete since the integral curves $\sigma_m(t) = U_t(m)$ are defined on $\mathbb{R}$ for all $m \in M$. Conversely, if L is a complete vector field with integral curves $\sigma_m(t)$, then it is an infinitesimal generator of a unique OPG of M defined by $U_t(m) = \sigma_m(t)$ $(m \in M)$. It follows that the correspondence between the set of all OPG's of M and the set of complete C^∞ vector fields on M is 1-1. We note that when M is compact, then every C^∞ vector field On M is complete [27, 28].

§ (1.6) The Differential of a Map

Let π be a differentiable map from M into M'. If $f \in C^\infty(O'_{\pi(m)})$, where $\pi(O_m) \subseteq O'_{\pi(m)}$, then $\pi^* f \in C^\infty(O_m)$. For every $v \in T_m$, set

$$((\pi_*)v)(f) = v(\pi^* f). \tag{1.1.9}$$

It is easily verified that $(\pi_*)_m v$ is a tangent vector on M' at $\pi(m)$. Thus $(\pi_*)_m$ is a map from T_m into $T_{\pi(m)}$. Also, this map is linear. $(\pi_*)_m$ is called the differential of the differentiable map π at m.

Let $(x^1, \ldots, x^N)$ and $(y^1, \ldots, y^N)$ be local coordinate systems on O_m and $O'_{\pi(m)}$ respectively and set $\pi^* y^i = \pi^i$. Then we have

$$(\pi_*)_m \left(\frac{\partial}{\partial x^i}\right)_m = \frac{\partial \pi^j}{\partial x^i}(m) \left(\frac{\partial}{\partial y^j}\right)_{\pi(m)} . \tag{1.1.10}$$

Thus the matrix of (π_*) is

$$(\pi_*)_{ij} = \left(\frac{\partial \pi^i}{\partial x^j}\right) . \tag{1.1.11}$$

§ 2 Tensor Analysis on Manifolds

We have seen that the tangent and cotangent spaces of a manifold M possess a vector space structure. The entities dealt with in this section are constructed over a vector space T. However, and without spelling it out, T will refer to a tangent space at some point of the manifold M.

§ (2.1) The Natural Embedding of T into T**

Let T be a vector space. If $w \in T^*$, then w is a function on T. Thus wv is a function of the T-valued variable v. We can take a contrary view, and consider instead wv as a function of the T*-valued variable w with value wv. Adopting this last view, v becomes a linear functional on T*, and hence it belongs to T**. Though the variable $v \in T$ and the functional $v \in T^{**}$ are not really the same, we ignore this difference and identify T with T**. Such identification is called the natural embedding of T into T**. When T is finite dimensional, as it is in our case, the natural embedding of T into T** is an isomorphism between T and T** [27].

We can adopt a third point of view and consider wv as a bilinear functional $(\ \ |\ \ ): V^* \times V \to \mathbb{R}$, defined by $(w|v) = wv$. If $w = w_i dx^i$ and $v = v^j \frac{\partial}{\partial x^j}$, then $(w|v) = w_i v^j \ (dx^i | \frac{\partial}{\partial x^j}) = w_i v^i$.

§ (2.2) Tensors, Tensor Spaces and Tensor Fields on a Manifold

A multilinear functional A with variables all either in T or in T* is called a tensor over T. Let A be a tensor over T of the form

$$A : \underbrace{T^* \times \cdots \times T^*}_{r\text{-times}} \times \underbrace{T \times \cdots \times T}_{s\text{-times}} \to \mathbb{R}$$

The set of all such tensors forms a real vector space called the tensor space over T; it is denoted by any of the following symbols

$$T^r_s \equiv T(r,s) \equiv \underbrace{T \otimes \cdots \otimes T}_{r\text{-times}} \otimes \underbrace{T^* \otimes \cdots \otimes T^*}_{s\text{-times}}.$$

The numbers r and s are called the type numbers or orders of the tensor, wit r as the contravariant order, and s as the covariant order.

Let M be a C^{∞} manifold. A tensor field of the type (r,s) on M is an assignment of a tensor of the type (r,s) to each point of M. If T is such tensor field on M, then T is said to be C^{∞} if $T(W_1,\ldots, W_r, L_1,\ldots, L_s)$ is C^{∞} for all C^{∞} contravarient vector fields $L_i (i = 1,\ldots, s)$ and all C^{∞} covariant vector fields $W_j (j = 1,\ldots, r)$.

If $A \in T^r_s$ and $B, C \in T^u_t$, then we can define the tensor product $A \otimes B \in T^{r+u}_{s+t}$ by

$$A\otimes B(\omega_1,\ldots,\omega_{r+u}, v^1,\ldots v^{s+t}) = A(\omega_1,\ldots,\omega_r, v^1,\ldots,v^s)B(\omega^{r+1},\ldots,\omega^{r+u}, v_{s+1},\ldots,v_s$$

It can be verified that

$$\begin{aligned} (A\otimes B)\otimes C &= A\otimes(B\otimes C) \,, \\ A\otimes(B+C) &= A\otimes B + A\otimes C \,, \\ (B+C)\otimes A &= B\otimes A + C\otimes A \,. \end{aligned} \tag{1.2.1}$$

§ (2.3) Coordinate Expressions of Tensors and Transformation Laws

Theorem: The N^{r+s} tensors

$$\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_s}, \tag{1.2.2}$$

$(i_1,\ldots, i_r; j_1,\ldots, j_s = 1,\ldots, N)$ form a basis of the tensor space T^r_s [27-29].

In view of the above theorem, every element $A \in T^r_s$ may be written as

$$A = A^{i_1\cdots i_r}_{j_1\cdots j_s}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_s}, \tag{1.2.3}$$

We call the set of n^{r+s} numbers $A^{i_1\ldots i_r}_{j_1\ldots j_s} = A(dx^{i_1},\ldots, dx^{i_r}, \frac{\partial}{\partial x^{j_1}},\ldots, \frac{\partial}{\partial x^{j_s}}$ The components of the tensor with respect to the basis $\{\frac{\partial}{\partial x^i}\ ;\ i = 1,\ldots, N\}$ in T. From (1.2.3) we deduce that the components of a tensor $A \in T^r_s$ transform under a basis change according to the law

$$A'^{i_1 \cdots i_r}_{j_1 \cdots j_s} = A^{\ell_1 \cdots \ell_r}_{k_1 \cdots k_s} \frac{\partial x'^{i_1}}{\partial x^{\ell_1}} \cdots \frac{\partial x'^{i_r}}{\partial x^{\ell_1}} \frac{\partial x^{k_1}}{\partial x'^{j_1}} \cdots \frac{\partial x^{k_s}}{\partial x'^{j_s}}. \qquad (1.2.4)$$

§ (2.4) Symmetric and Anti-Symmetric Tensors

A tensor A is symmetric (anti-symmetric) in the ith and in the jth covariant indices if its components with respect to every basis are unchanged (change sign) when these indices are interchanged. The following conditions on a tensor A are equivalent.

(i) A is symmetric (anti-symmetric) in the ith and jth covariant indices.

(ii) A is symmetric (anti-symmetric) multilinear functional in the ith and jth covariant variables.

(iii) The components of A with respect to any basis are unchanged (change sign) when the ith and jth indices are interchanged.

Symmetry and anti-symmetry with respect to a pair of contravariant indices are defined similarly. A tensor is said to be covariant symmetric (anti-symmetric) if it is symmetric (anti-symmetric) in each pair of its covariant indices. Contravariant symmetric (anti-symmetric) tensors are defined in a similar fashion.

§ (2.5) Covariant Tensors of Type (0,2)

From the definition of a tensor, a covariant tensor of type (0,2) is a bilinear functional $b: T \times T \to \mathbb{R}$. The set $\{dx^i \otimes dx^j;\ i,j = 1,\dots, N\}$ is a basis of T^0_2. In terms of this basis every element $b \in T^0_2$ could be written as $b = b_{ij}\, dx^i \otimes dx^j$. It is clear that b is symmetric if $(b_{ij}) = (b_{ji})$ and is anti-symmetric if $(b_{ij}) = -(b_{ji})$.

Let $b \in T^0_2$. For a fixed element $v \in T$, $b(u,v)$ is a linear functional of $u \in T$, so it belongs to T^*. Denote this linear map by $\tilde{v} \in T^*$, so that

$$\tilde{v}(u) = (\tilde{v}\,|\,u) = b(u,v) \quad \forall\, u \in T. \tag{1.2.5}$$

since b is bilinear, $\tilde{v}$ is a linear function of v, and so we have

$$\sim\, : T \longrightarrow T^{*},\ v \longrightarrow \tilde{v}. \tag{1.2.6}$$

Conversely, if $\sim : T \to T^*$ is linear, then we can define a bilinear functional b: $T \times T \to \mathbb{R}$ by $b(u,v) = (\tilde{v}|u)$, that is a tensor of the type (0,2).

We may express $\sim$ in terms of coordinates as follows

$$\begin{aligned} b(u,v) &= b_{ij}\,(dx^i|u)(dx^j|v) \\ &= b_{ij}\,(dx^i|u)(dx^j|v^\ell\,\partial/\partial x^\ell) \\ &= (b_{ij}\,v^j\,dx^i|u) \\ \Rightarrow\ \tilde{v} &= (b_{ij}\,v^j)\,dx^i. \end{aligned} \tag{1.2.7}$$

In a similar fashion, we may interpret b as a linear function $_\sim : T \to T^*$ taking each $u \in T$ to $\underset{\sim}{u} \in T^*$ such that $b(u,v) = (\underset{\sim}{u}|v)\ \forall\, v \in T$. It is easily verified that

$$\underset{\sim}{u} = (b_{ij}\,u^i)\,dx^j. \tag{1.2.8}$$

If b is symmetric, then $\sim\, = {}_\sim$. If b is anti-symmetric, then $\sim\, = -{}_\sim$.

If the matrix (b_{ij}) is non-singular, then b is said to be non-degenerate. It can be shown [29] that: b is non-degenerate if $\sim$ has an inverse, or $_\sim$ has an inverse, or for every $u \in T$, $u \neq 0$, $\exists\, v \in T$ such that $b(u,v) \neq 0$. If b is non-singular, then the inverse map $\approx : T^* \to T$ such that $\overset{\approx}{v} = v$ has the matrix (b^{ir}), where $b^{ir}b_{ij} = \delta^r_j$.

Let W be any C^∞ covariant vector field in a C^∞ manifold M. For each pair L_1, L_2 of C^∞ contravariant vector fields, set

$$dW(L_1, L_2) = L_1(W(L_2)) - L_2(W(L_1)) - W([L_1, L_2]), \tag{1.2.9}$$

where $[L_1, L_2] = L_1L_2 - L_2L_1$ (it is called the commutator of L_1 and L_2). It can be verified that dW is an anti-symmetric covariant C^∞

tensor field of type (0.2)

If $W = df$ for some $f \in C^\infty(M)$, then

$$dW(L_1, L_2) = L_1(df(L_2)) - L_2(df(L_1)) - df([L_1, L_2])$$
$$= L_1(L_2(f)) - L_2(L_1(f)) - [L_1, L_2](f) = 0 .$$

Thus a necessary condition that $W = df$ for some f is that $dW = 0$. This condition is not sufficient but it is sufficient locally. This can be seen as follows. If $W = w_i dx^i$, then

$$(dW)_{ij} = dW\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{\partial w_j}{\partial x^i} - \frac{\partial w_i}{\partial x^j} . \qquad (1.2.10)$$

Thus $dW = 0$ iff $\frac{\partial w_j}{\partial x^i} = \frac{\partial w_i}{\partial x^j}$, and hence $w_i = \frac{\partial f}{\partial x^i}$ in some neighbourhood of every point.

The covariant vector field is said to be closed if $dW = 0$. If $W = df$ for some function f on M, then, as mentioned in §(1.4), W is said to be exact.

§ (2.6) Scalar Product, Riemannian Metric and Riemannian Manifolds

By a scalar product G_m on a vector space T_m is meant a symmetric positive-definite tensor of type (0,2). The scalar product is expressed in terms of the coordinates as $G_m = g_{ij}(m)\ (dx^i)_m \otimes (dx^j)_m$. If at every point m of a manifold M, a scalar product is defined on T_m such that $g_{ij} \in C^\infty(M)$, we say that a Riemannian metric is defined on M. A manifold M endowed with a Riemannian metric is called a Riemannian manifold,i.e. a Riemannian manifold is a C^∞ manifold with a C^∞ positive-definite symmetric second order covariant tensor field G.

Since G is positive definite, it is non-degenerate. Hence the matrix g_{ij} has an inverse g^{ij} such that $g_{ij}g^{ie} = \delta^e_j$. It can easily be proved that g^{ij} are the components of a symmetric contravariant tensor of type (2,0)

It follows from §(2.5) that the metric G on M induces an isomorphism between T_m and T_m^* at each $m \in M$. This isomorphism is defined by

$$v = v^i \partial/\partial x^i \longleftrightarrow \tilde{v} = p_j \, dx^j , \tag{(1.2.11a)}$$

where

$$p_j = g_{ij} v^i , \quad v^i = g^{ij} p_j . \tag{1.2.11b}$$

§ (2.7) Motions and Infinitesimal Motions of a Riemannian Manifold

Let M and M' be Riemannian manifolds with metrics G and G' respectively. If π is a differentiable map from M to M' such that

$$\| \pi_*(v) \| = \| v \| \quad \forall v \in T \ \ (T \in TM), \tag{1.2.12}$$

then π is called an isometry from M to M'. If a diffeomorphism π of a Riemannian manifold M onto itself is an isometry, then π is called a motion of M. The set of all motions on M form a group called the group of motions of M. For, if π and π' are motions of M, then $\| (\pi \, \pi')_* (v) \| = \| (\pi_* (\pi'_*(v)) \| = \| \pi'_*(v) \| = \| v \| \ \forall v \in T \ (T \in TM)$, and hence $\pi \, \pi'$ is a motion of M. If π'' is a motion of M as well, then it is clear $\pi(\pi' \pi'') = (\pi \, \pi')\pi''$. The identity element of the group is the identity map $I : M \to M$. Also, for every motion π of M, there exists an inverse element, namely $\pi^{-1}: M \to M$, which is a motion of M.

Let L be a vector field on M. If for arbitrary vector fields Y, Z on M, the relation

$$L(G(Y,Z)) = G([L,Y],Z) + G(Y,[L,Z]) \tag{1.2.13}$$

holds, then L is called an infinitesimal motion of M or a Killing vector field on M. Let $(x^1, \ldots, x^N)$ be some local coordinate system in M. The vector field L can be expressed locally as $L = \xi^j(x^1, \ldots, x^N)\partial/\partial x^j$. Setting $Y = \partial/\partial x^n$, $Z = \partial/\partial x^m$ in (1.2.13), we obtain the system of partial differential equations

$$g_{mj}\xi^{j}_{,n} + g_{nj}\xi^{j}_{,m} + g_{mn,j}\xi^{j} = 0 \quad (m,n=1,\cdots,N). \qquad (1.2.14)$$

These equations are called the Killing equations. The vector field $L = \xi^j \partial/\partial x^j$ is an infinitesimal motion iff its components ξ^j satisfy the Killing equations (1.2.14). It is easy to see that the Killing equations can be written in the alternative form

$$g^{mj}\xi^{n}_{,j} + g^{nj}\xi^{m}_{,j} - g^{mn}_{,j}\xi^{j} = 0 \quad (m,n=1,\cdots,N) \qquad (1.2.15)$$

The following theorem bridges the gap between the concepts of an infinitesimal motion and a motion of a Riemannian manifold:

Theorem: Let L be a complete vector field in M. The vector field L is an infinitesimal motion of M iff each element of the OPG generated by L is a motion of M.

§ 3 Classical Mechanics

Perhaps the most basic law of classical particle mechanics is that the future coordinates and velocities of a system of particles are uniquely determined by the coordinates and velocities of the system at some instant t_o.

Let M be a Riemannian manifold and $(x^1, \ldots, x^N) = \underline{x}$ be some local chart in M. If $(v^1, \ldots, v^N) = \underline{v}$ is the time derivative of $\underline{x}$, then the numerical value of the 2N-multiplet $(\underline{x},\underline{v}) \in TM$ at some instant determines the state of the system (i.e., its coordinates and velocities) at any time. The tangent bundle TM is the set of all possible values of the states $(\underline{x},\underline{v})$; it is called the state space.

§ (3.1) The Dynamical Group of a System of Particles

We may re-express the previous ideas as follows. The time evolution of the system may be described by an OPG U_t of the state space so that the state β at time t is given by $U_t(\beta_o)$, where β_o is the state at t = 0. The OPG U_t is called the dynamical group of the system. As explained in § 1.5 (2.4), U_t defines a unique vector field on TM, namely its infinitesimal generator $L = \xi_o^i(\underline{x},\underline{v})\,\partial/\partial x^i + \xi^i(\underline{x},\underline{v})\,\partial/\partial v^i$. The orbits of U_t are curves in TM satisfying the following system of ordinary differential equations:

$$\frac{dx^i}{dt} = \xi_o^i(\underline{x},\underline{v}) \;, \quad \frac{dv^i}{dt} = \xi^i(\underline{x},\underline{v}) \tag{1,3,1}$$

In this system, the x^i's and the v^i's are considered as functions of time and the initial values of the coordinates and velocities. Since $dx^i/dt = v^i$, all the functions $\xi_o^i(\underline{x},\underline{v})$ are known, and hence we have

$$\frac{dx^i}{dt} = v^i \;, \quad \frac{dv^i}{dt} = \xi^i(\underline{x},\underline{v}). \tag{1,3,2}$$

§ (3.2) The Cotangent Bundle as the State Space

Via the metric tensor G we can set up a 1-1 map from the state space TM onto the cotangent bundle T*M. This map is defined by $(m,v) \to (m,\tilde{v})$, where $m \in M$ and $v \in T_m$. By (1.2.11), the state is specified by the 2N-multiplet $(x^1,\ldots, x^N, p_1,\ldots, p_N)$, where $p_i = g_{ij}v^j$. Thus the cotangent bundle may serve as the state space. The dynamical group U_t is considered to be acting on T*M. If $L = \xi_o^i(\underline{x},\underline{p})\partial/\partial x^i + \xi^i(\underline{x},\underline{p})\partial/\partial p_i$ is the infinitesimal generator of U_t, then the orbits of U_t are curves in T*M satisfying the following system of ordinary differential equations:

$$\frac{dx^i}{dt} = \xi_o^i(\underline{x},\underline{p}) \ , \ \frac{dp_i}{dt} = \xi^i(\underline{x},\underline{p}) .$$

If $f(\underline{x},\underline{p})$ is a differentiable function defined on T*M, then the rate of change of f along the orbits of U_t is given by

$$\frac{d}{dt}[f(U_t(\underline{x},\underline{p})] = \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt} = \frac{\partial f}{\partial x^i}\xi_o^i + \frac{\partial f}{\partial p_i}\xi^i . \quad (1.3.3)$$

f is said to be an integral of our dynamical system iff it is constant on the orbits of U_t. It follows from (1.3.3) that f is an integral iff this latter expression vanishes identically.

Suppose that $L = \eta_o^i\ \partial/\partial x^i + \eta^i\ \partial/\partial p_i$ is the infinitesimal generator of an OPG V_t of T*M. If

$$\eta_o^i = -\frac{\partial f}{\partial p_i} \ , \ \eta^i = \frac{\partial f}{\partial x^i} \ , \quad (1.3.4)$$

where $f \in C^\infty(T^*M)$, then it is obvious that f is constant on the orbits of V_t. The OPG V_t in this case is called an OPG of contact transformations its infinitesimal generator L is called infinitesimal contact transformations. The function f determines V_t uniquely, and is uniquely determined by V_t up to an additive constant. f is called the fundamental invariant of V_t.

If V_t is an OPG of contact transformations whose fundamental invariant is f, then a function $g \in C^\infty(T^*M)$ is constant on the V_t orbits iff

$$-\frac{\partial g}{\partial x^i}\frac{\partial f}{\partial p_i} + \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial x^i} = 0 . \tag{1.3.5}$$

The above expression is called the Poisson bracket of f and g and is denoted by $\{f,g\}$. It is clear that $\{f,g\} = -\{g,f\}$, and hence $\{f,g\} = 0$ iff $\{g,f\} = 0$. Thus g is constant on the orbits of the OPG of contact transformations defined by f iff f is constant on the orbits of the OPG of contact transformations defined by g.

§ (3.3) The fundamental Covariant Vector Field and Classification of Covariant Vector Fields

Consider the map $\pi: O \subseteq T^*M \to M$ defined by $\pi(\underline{x},\underline{p}) = \underline{x}$. For every element $(\underline{x},\underline{p}) \in T^*M$, $(\pi_*)_{(\underline{x},\underline{p})}$ is a linear map of $T_{(\underline{x},\underline{p})}$ onto $T_{\underline{x}}$. The adjoint map $(\pi_*)^+_{(\underline{x},\underline{p})}$ is a linear map of $T^*_{\underline{x}}$ onto $T^*_{(\underline{x},\underline{p})}$. Thus we have an assignment of an element of $T^*_{(\underline{x},\underline{p})}$ (the cotangent space of T^*M at $(\underline{x},\underline{p})$) to each $(\underline{x},\underline{p}) \in T^*M$ that is a covariant vector field W^o. This vector field is called the fundamental covariant vector field of the manifold. A general covariant vector field W on $O \subseteq T^*M$ can be expressed as $W = a_i(\underline{x},\underline{p})\, dx^i + b_i(\underline{x},\underline{p})\, dp_i$. Let us calculate the a's and the b's of W^o. Since $\pi(\underline{x},\underline{p}) = \underline{x}$, by (1.1.11), we have

$$(\pi_*)_{ij} = \overset{\longleftarrow\ 2N\ \longrightarrow}{\left[\begin{array}{cccc:c} 1 & & 0 & & \\ & 1 & & & \\ & & 1 & & 0 \\ 0 & & \ddots & & \\ & & & 1 & \end{array}\right]} \; \updownarrow N$$

The matrix of $(\pi_*)^+$ is the transpose of this. Thus $(\pi_*)^+_{(\underline{x},\underline{p})} p = (\underline{p},\underline{o})$, and hence

$$W^o = p_i\, dx^i . \tag{1.3.6}$$

dW^o is an anti-symmetric second order covariant tensor field. By (1.2.10) we have

$$dW^o = dp_i \otimes dx^i - dx^i \otimes dp_i , \qquad (1.3.7)$$

which is non-degenerate everywhere. Let L_1 and L be C^∞ contravariant vector fields on M. The map $L \to \tilde{L}$, where $\tilde{L}(L_1) = dW^o(L_1,L)$ for all such L_1 is 1-1 onto map from C^∞ contravariant vector fields in T*M to the C^∞ covariant vector fields in T*M. If

$$L = a_i(\underline{x}, \underline{p})\partial/\partial x^i + b_i(\underline{x}, \underline{p})\partial/\partial p_i , \qquad (1.3.8)$$

then it can be easily verified that

$$\tilde{L} = -b_i(\underline{x}, \underline{p})dx^i + a_i(\underline{x}, \underline{p})dp_i . \qquad (1.3.9)$$

We note that the map $\underset{\sim}{}$ produces a difference in sign from $\tilde{}$, and hence $\tilde{L} = -\underset{\sim}{L}$. Consistently with §(2.5), we use the tilde for the inverse of $\tilde{}$ so that $\tilde{\tilde{L}} = L$. If f_1, $f_2 \in C^\infty(T^*M)$, then

$$dW^o(\tilde{df_1}, \tilde{df_2}) = \frac{\partial f_2}{\partial x^i}\frac{\partial f_1}{\partial p_i} - \frac{\partial f_2}{\partial p_i}\frac{\partial f_1}{\partial x^i} = \{f_2, f_1\} \qquad (1.3.10)$$

A contravariant vector field L in T*M is called globally Hamiltonian if $\tilde{L}$ is exact, i.e., there exists $F \in C^\infty(T^*M)$, such that $L = \tilde{dF}$. L is called locally Hamiltonian if $\tilde{L}$ is closed, i.e. $d\tilde{L} = 0$. In the second case, $L = \tilde{dF}$ for some $F \in C^\infty(M)$ in some neighbourhood of each point. In both cases the differential equations defined by L have the form

$$\frac{dx^i}{dt} = \frac{\partial F}{\partial p_i} , \quad \frac{dp_i}{dt} = -\frac{\partial F}{\partial x^i} . \qquad (1.3.11)$$

Let $L_1 = \tilde{df}_1$ and $L_2 = \tilde{df}_2$ be globally Hamiltonian infinitesimal generators of the OPG's of T*M, U_t^1 and U_t^2 respectively. The rate of change of f_2 along the orbits of U_t^1 is

$$L_1(f_2) = \tilde{df_1}(f_2) = \{f_1, f_2\} = dW^o(\tilde{df_2}, \tilde{df_1}). \qquad (1.3.12)$$

§ (3.4) The Basic Assumption of the Hamiltonian Mechanics

The fundamental assumption of the Hamiltonian mechanics is that the dynamical group U_t of T*M is generated by a globally Hamiltonian vector field L.

Let $L = d\tilde{H}$, where $H \in C^\infty(T^*M)$. Since $dx^i/dt = v^i$, by (1.2.11) we have

$$\frac{dx^i}{dt} = g^{ij} p_j .$$

Since also $dx^i/dt = \partial H/\partial p_i$, we have

$$\frac{\partial H}{\partial p^i} = g^{ij} p_j \tag{1.3.13}$$

Thus

$$H = \frac{1}{2} g^{ij} p_i p_j + Q(\underline{x}) , \tag{1.3.14}$$

where $Q(\underline{x}) \in C^\infty(M)$, and it is uniquely determined by $H(\underline{x},\underline{o}) = Q(\underline{x})$. Thus, given G and $Q(\underline{x})$, H is uniquely determined, and hence so also is the motion of the system. The function H determines g^{ij} and hence the Riemannian metric G of M. H is called the Hamiltonian of the system.

§ (3.5) Physical Observables

When the Riemannian metric G is used to map TM onto T*M so that the latter may be regarded as the state space, we refer to T*M as the phase space. An observable or a dynamical variable is by definition a real-valued function on the phase space. The observables $Q(\underline{x})$, $\frac{1}{2} g^{ij}p_ip_j$, and $\frac{1}{2} g^{ij}p_ip_j + Q(\underline{x})$ are called respectively the potential energy, the kinetic energy and the total energy of the system. The momentum observable is defined as follows. Let V_t be an OPG of M whose infinitesimal generator is $L = \xi^i\partial/\partial x^i$. L generates a function P on T*M by

$$P(\underline{x}(m), \underline{p}(m)) = (p_i \, dx^i | \xi^i \partial/\partial x^i) = \xi^i p_i . \tag{1.3.15}$$

The function P is called a momentum observable. Functions in T*M linear on each T_m^* are called generalized momenta.

An observable F is a constant of motion if it is constant on the orbits of the dynamical group U_t of T*M, i.e., $\{H,F\} = 0$. A momentum observable $P = \xi^i p_i$ is a constant of the free motion iff

$$\{ \xi^i p_i , g^{mn} p_m p_n \} = 0$$
$$\Longleftrightarrow (g^{mj} \xi^n_{,j} + g^{jn} \xi^m_{,j} - g^{mn}_{,j}) p_m p_n = 0 .$$

Thus P is a constant of the free motion iff the above expression vanishes identically, and hence, iff

$$g^{mj} \xi^n_{,j} + g^{jn} \xi^m_{,j} - g^{mn}_{,j} \xi^j = 0 .$$

But these are just the Killing equations (1.2.15). Thus P is a constant of the free motion iff $\xi^i \partial/\partial x^i$ is a Killing vector field.

CHAPTER II

Critique of the Canonical Quantization Scheme

§1. The Canonical Quantization Scheme

In the Hamiltonian formulation of classical mechanics, the generalized momenta p_i are treated as independent variables on a par with the spatial coordinates x^i. The variables x^i and p_i are called canonical variables.

The method of canonical quantization is based on the assumption that associated with the variables x^i, p_i obeying the usual Poisson brackets relations

$$\begin{aligned} &\{x^i, p_k\} = \delta^i_k \,, \\ &\{x^i, x^k\} = \{p_i, p_k\} = 0 \,, \end{aligned} \tag{2.1.1}$$

there correspond quantum mechanical operators $\hat{x}^i$, $\hat{p}_i$ satisfying the commutation relations [14-17, 19-21]

$$\begin{aligned} &[\hat{x}^i, \hat{p}_k] = i\hbar\, \delta^i_k \,, \\ &[\hat{x}^i, \hat{x}^k] = [\hat{p}_i, \hat{p}_k] = 0 \,. \end{aligned} \tag{2.1.2}$$

Moreover, since every classical quantity is a function of the canonical variables, the corresponding quantum mechanical operator is obtained by replacing the canonical variables by their corresponding quantum mechanical operators [14-17]. The order of the canonical variables in a classical observable is immaterial. However, when the corresponding quantum observable is constructed, special care should be taken in ordering the canonical operators such that the resulting operator is self-adjoint.

The basic scheme of the canonical quantization is to find $\hat{p}_i$ explicitly, then to express the Hamiltonian in terms of $\hat{x}^i$ and $\hat{p}_i$. Once this is achieved one proceeds to establish the equations of motion for observables in the Heisenberg picture in the usual way.

Subjects related to canonical quantization methods are discussed in many papers [12, 13, 19-23] and textbooks [14-18, 24]. Though their approaches to the problem differ, most of the aforementioned textbooks and papers implement the above scheme partially or totally.

A generally adopted expression for $\hat{p}_i$ is [19-22]

$$\hat{p}_i = -i\hbar(\partial/\partial x^i + \tfrac{1}{2}(\ln\sqrt{g})_{,i}), \tag{2.1.3}$$

where g is the determinant formed by the metric g_{ij} and the comma denotes differentiation with respect to a coordinate. Another way of writing the above expression is

$$\hat{p}_i = -i\hbar(\partial/\partial x^i + \tfrac{1}{2}\,\mathrm{div}(\partial/\partial x^i)) \,. \tag{2.1.4}$$

Sometimes $\mathrm{div}(\partial/\partial x^i)$ is expressed in terms of Christoffel's symbols of the second type [38, p120], giving rise to the following form:

$$\hat{p}_i = -i\hbar(\partial/\partial x^i + \tfrac{1}{2}\left\{ {k \atop ik} \right\}) \,. \tag{2.1.5}$$

In this chapter we shall raise specific criticisms of the canonical quantization scheme. Instead of the canonical momenta we will adopt the momenta prescribed by Mackey [4] and discussed in chapter III. Mackey's approach will be shown in the next chapter to produce a rigorous basis for quantization in spaces of constant curvature.

§2. A Critique of the Scheme

There are two major objections. The first is that given any classical variables x^i, p_i we may not be able to establish corresponding quantum observables. The second objection is to the choice to coordinates themselves.

§(2.1). The Momenta

Consider a 1-dimensional manifold M coverable by a single coordinate chart x. Let the metric form be $ds^2 = g(x)\, dx^2$. The usual quantization scheme leads to the operator [19-22]

$$\hat{p} = -i\hbar\,(d/dx + \tfrac{1}{2}\,(\ln\sqrt{g}\,)_{,x}) \tag{2.2.1a}$$

$$\equiv -i\hbar\,(d/dx + \tfrac{1}{2}\,\mathrm{div}\,(d/dx)) \tag{2.2.1b}$$

as the quantum momentum operator. However, this is not satisfactory. The operator $\hat{p}$ as it stands is not even defined since its domain is not specified. A reasonable way to inject some mathematical rigour into this is to proceed as follows: Take the domain of $\hat{p}$ to be the set

$$D_p = \{\,\psi \mid \psi \in C^1_0(M);\ \psi,\ p\psi \in L^2(M)\}\ , \tag{2.2.2}$$

where $C^1_0(M)$ denotes the set of absolutely continuous functions with compact support on M, and $L^2(M)$ denotes the set of square integrable complex functions on M. The operator $\hat{p}$ with domain D_p is then well-defined. The properties of $\hat{p}$ are characterized by the following two propositions whose proofs may be found in [App. 1].

Proposition (2.1)

1. $\hat{p}$ is symmetric.
2. The adjoint of $\hat{p}$ is

$$\hat{p}^{+} = -i\hbar(d/dx + \tfrac{1}{2}(\ln\sqrt{g}\,)_{,x}) \tag{2.2.3}$$

with the domain

$$D_{p^+} = \{\,\psi \mid \psi \in C^1(M);\ \psi,\ \hat{p}^{+}\psi \in L^2(M)\}\ , \tag{2.2.4}$$

3. $\hat{p}$ is not self-adjoint.

Proposition (2.2)

1. If the range of x is (0,b), where b is finite, then

   (i) $\hat{p}$ is not essentially self-adjoint,

   (ii) consequently, $\hat{p}$ has no unique self-adjoint extension, though self-adjoint extensions do exist. A typical extension is

   $$\hat{P} = -i\hbar\left(d/dx + \frac{1}{2}(\ln\sqrt{g})_{,x}\right) \tag{2.2.5}$$

   with the domain

   $$D_P = \{\psi \mid \psi \in C^1(M);\ \psi, P\psi \in L^2(M);\ e^{i\beta}\psi(b)\sqrt[4]{g(b)} = \psi(0)\sqrt[4]{g(0)}\}, \tag{2.2.6}$$

   where β is fixed in the interval $[0, 2\pi)$. $\hat{P}$ is not unique in the sense that different values of β lead to different operators,

   (iii) $\hat{P}$ has a purely discrete spectrum

   $$\lambda_n = (2n\pi - \beta)\hbar/b \quad (n \in N) \tag{2.2.7}$$

   corresponding to the eigenfunctions

   $$\psi_n(x) = e^{i(2n\pi - \beta)x/b} / \sqrt[4]{b^2 g}\,. \tag{2.2.8}$$

2. If the range of x is $(0, \infty)$, then

   (i) $\hat{p}$ is not essentially self-adjoint,

   (ii) $\hat{p}$ has no self-adjoint extension.

3. If the range of x is $(-\infty, \infty)$, then $\hat{p}$ is essentially self-adjoint with a unique self-adjoint extension $\hat{P} = \hat{p}^+$.

The importance of essential self-adjointness of $\hat{p}$ lies in the existence of a unique self-adjoint extension of $\hat{p}$ which may serve as the quantum mechanical momentum without ambiguity. If a self-adjoint extension is not unique, then quite apart from the ambiguity of any particular choice, there are serious physical difficulties. These will be presented explicitly soon.

Example 1. $M = (0,\infty)$ with metric $ds^2 = dx^2$.

Since $g = 1$, the momentum operator $\hat{p}$ takes the form $-i\hbar d/dx$. According to proposition (2.2), $\hat{p}$ is not essentially self-adjoint and it has no self-adjoint extension. In confirmation of this deduction we refer to [1,3].

Example 2. $M = E^2$ with polar coordinates (r,ϕ).

From the metric form

$$ds^2 = dr^2 + r^2 d\phi^2,$$

the formal momentum operator is, according to (2.1.3),

$$p_r = -i\hbar(\partial/\partial r + \tfrac{1}{2}(\ln\sqrt{g})_{,r}) = -i\hbar(\partial/\partial r + 1/2r). \tag{2.2.9}$$

As proposition (2.2) shows, $\hat{p}_r$ is not essentially self-adjoint and it has no self-adjoint extension. In this context we note that in classical mechanics the canonical variables (r,ϕ,p_r,p_ϕ) and (x,y,p_x,p_y), where (x,y) is a rectangular Cartesian coordinate system in E^2, are canonically equivalent. They are related by the point transformation $x = r\cos\phi$, $y = r\sin\phi$; the corresponding canonical transformation may be obtained from the generating function $u = r\cos\phi\, p_x + r\sin\phi\, p_y$. In fact, all point transformations are canonical [31]. However, the operators $\hat{r}$, $\hat{\phi}$, $\hat{p}_r$, $\hat{p}_\phi$ are not unitarily equivalent to $\hat{x}$, $\hat{y}$, $\hat{p}_x$, $\hat{p}_y$ [16]. In other words, there is no unitary operator $\hat{U}$ such that $\hat{r} = \hat{U}\hat{x}\hat{U}^+$, etc. For we know that a unitary transformation preserves the self-adjointness and the spectrum of an operator [25], and hence if such a transformation exists, then $\hat{p}_r$ has to be self-adjoint and $\hat{r}$ has to possess a continuous spectrum ranging from $-\infty$ to $+\infty$. But both conclusions are incorrect; $\hat{p}_r$ is not self-adjoint [proposition (2.2)], and $\hat{r}$ cannot have negative spectrum according to its definition.

Example 3. The 2-sphere S^2.

Using polar coordinates and assuming the radius is equal to 1, the metric in S^2 takes the form

$$ds^2 = \sin^2\theta\, d\phi^2 + d\theta^2 \quad (0<\theta<\pi). \tag{2.2.10}$$

The canonical quantization renders

$$\hat{p}_\theta = -i\hbar(\partial/\partial\theta + \frac{1}{2}\cot\theta) \ [20].$$

By proposition (2.2), $\hat{p}_\theta$ has self-adjoint extensions $\hat{P}_\theta$ with periodic boundary conditions. Taking β in (2.2.6) to be equal to zero, the boundary conditions take the form

$$f(\pi)\sqrt[4]{g(\pi)} = f(0)\sqrt[4]{g(0)} \ .$$

$\hat{P}_\theta$ has a discrete spectrum $\{\lambda_n = 2n\hbar\}$, The corresponding normalized eigenfunctions are

$$\psi_n = \frac{e^{i2n\theta}}{\sqrt{2\pi^2 \sin\theta}} \ .$$

The Hamiltonian (we set the mass m = 1) is given by

$$\hat{H} = -\frac{\hbar^2}{2}\nabla^2 = -\frac{\hbar^2}{2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right).$$

Let us calculate the expectation value of $\hat{H}$ immediately after a momentum measurement of $\hat{P}_\theta$. This calculation gives

$$(\psi_n, \hat{H}\psi_n) = (4n^2 - \frac{1}{4})\frac{\hbar^2}{2} - \frac{\hbar^2}{8}(\psi_n, \frac{1}{\sin^2\theta}\psi_n).$$

But

$$(\psi_n, \frac{1}{\sin^2\theta}\psi_n) = \frac{1}{\pi}\int_0^\pi \frac{d\theta}{\sin^2\theta} = -\frac{1}{\pi}\cot\theta\Big|_0^\pi = \infty.$$

Mathematically this means simply that eigenfunctions of $\hat{P}_\theta$ are not in the domain of $\hat{H}$. The physical implications are serious, that is, after a $\hat{P}_\theta$-measurement, the system will be in a state of infinite energy expectation value . Such a situation cannot be physically meaningful.

§(2.2) The Coordinate Variables.

When we discussed the momentum operator it was explicitly found that the coordinate variable sometimes is a cause of trouble, since depending on its range we may or may not have a well-defined canonically conjugate momentum.

We know that not all differentiable manifolds may be covered by a single coordinate chart. The circle S^1 is a simple example. S^1 is not homeomorphic to an open interval of the real line. Two overlapping charts are required to cover S^1. The polar angle θ could be used as a local coordinate chart and the two charts

$$\theta_1 \in (\delta, 2\pi - \delta) \ , \ \theta_2 \in (\pi + \delta, 3\pi - \delta) ,$$

where δ is a small positive number, form an atlas on S^1. The metric is equal to a constant everywhere. S^1 is what one calls a parallizable manifold [27], where it is possible to define a C^∞ vector field which is nowhere zero. In S^1 the vector field L defined by

$$\begin{aligned} L &= d/d\theta_1 \qquad \theta_1 \in (\delta, 2\pi - \delta) \\ &= d/d\theta_2 \qquad \theta_2 \in (\pi + \delta, 3\pi - \delta) \end{aligned}$$

is such a global vector field. The operator $\hat{P} = -i\hbar L$ with domain of definition D_P formed by the set of absolutely continuous functions ψ on S^1 is self-adjoint. Thus $\hat{P}$ serves as a momentum operator. The question now is whether we can regard $\hat{\theta}_1, \hat{P}$ or $\hat{\theta}_2, \hat{P}$ as canonical variables. The answer is no. Simply, θ_1 (or θ_2) is not defined throughout the manifold S^1. This means that we do not have a self-adjoint observable associated with θ as it stands [4]. A natural extension of θ to the entire circle is the function ϕ which takes values in the range $[0, 2\pi)$. The coordinate operator $\hat{\phi}$ defined by $\{\hat{\phi}\psi(\phi) = \phi\psi(\phi)$;

$\psi(\phi) \in L^2(S^1); 0 \leqslant \phi < 2\pi\}$ is properly defined quantum mechanically, but it is discontinuous, Hence a difficulty arises if we consider, say, the uncertainty relation between $\hat{P}$ and $\hat{\phi}$. The proplem of canonically conjugate momentum and coordinate on S^1 is essentially the same as the azimuthal angle and angular momentum. (A lot of controversy concerning the range of the coordinate variable θ and the related uncertainty relation is found in [5-11]).

§3. Digression from Canonical Quantization.

We have found in §2 that the only case for which one can obtain a well-defined unique pair of canonically conjugate quantum variables $\hat{x},\hat{p}$ is when x ranges from $-\infty$ to $+\infty$. When constructing such a pair in an arbitrary coordinate x, the difficulties arising are due to the coordinate system rather than to the manifold itself. Consider for instance the manifold $\mathbb{R}$ with a coordinate system x, $(-\infty < x < \infty)$. Let $ds^2 = dx^2$ be the metric form of $\mathbb{R}$. The operator $\hat{p} = -i\hbar d/dx$ is essentially self-adjoint with a unique self-adjoint extension $\hat{P}$, where $D_P = \{\psi | \psi \in C^1(M);\ \psi, \hat{P}\psi \in L^2(M)\}$. Now, if $\mathbb{R}$ is mapped onto the interval $(-\frac{\pi}{2},\frac{\pi}{2})$ via the map $x \rightarrow \tan y$, then the resultant canonical momentum $\hat{p}_y$ is given by

$$\hat{p}_y = -i\hbar(d/dy + \tan y).$$

$\hat{p}_y$ is not essentially self-adjoint. Thus $\hat{y}, \hat{p}_y$ cannot be taken as canonically conjugate quantum variables.

Conversely, the canonical momentum $\hat{p} = -i\hbar d/dx$ in the manifold $M=(0,b)$ with metric form $ds^2 = dx^2$ is not essentially self-adjoint. But if we perform the coordinate transformation $y = \tan(\pi x/b - \pi/2)$, then the operator

$$\hat{p}_y = -i\hbar(d/dy - 1/(1+y^2)),$$

arising from applying the canonical quantization scheme, is essentially self-adjoint with a unique self-adjoint extension $\hat{P}_y = \hat{p}_y^+$. It follows that we can choose in this manifold a coordinate system $-\infty < y < \infty$ so that $\hat{y}, \hat{p}_y$ constitute a well-defined pair of canonical variables.

Let us write (2.2.1) in the form

$$\hat{p} = -i\hbar(L + \frac{1}{2}\operatorname{div} L), \tag{2.3.1}$$

where $L = d/dx$. It follows that $\hat{p}$ is essentially self-adjoint only when L is a complete vector field. Assuming L is so, then on transforming to another coordinate y, eq. (2.3.1) may be written in the form

$$\hat{p} = -i\hbar\,(L + \frac{1}{2}\,\mathrm{div}\,L)$$

$$= -i\hbar\,(\frac{dy}{dx}\,\frac{d}{dy} + \frac{1}{2}\,\mathrm{div}(\frac{dy}{dx}\,\frac{d}{dy})). \tag{2.3.2}$$

Obviously $\hat{p}$ as given by (2.3.2) is essentially self-adjoint; it is simply the same $\hat{p}$ in (2.3.1) expressed in another coordinate system.

From the content of this last paragraph, one may anticipate that the vital point in establishing an essentially self-adjoint momentum operator is to relate it to a complete vector field in M, i.e. to an OPG of M. Contrary to the canonical momentum which originates from a particular choice of coordinate system, the momentum $\hat{p}$ given by (2.3.1) is intrinsically meaningful since it has nothing to do with the coordinate system employed. It must be indicated that having related the momentum operator to a complete vector field in M, we have abandoned the canonical quantization scheme characterized by the assumed commutation relations (2.1.2). These relations are no longer satisfied since

$$[x, -ih\,(L + \frac{1}{2}\,\mathrm{div}\,L)] = i\hbar\,\xi(x),$$

where $L = \xi(x)d/dx$.

The operator

$$\hat{P}_0 = -ih\,(L + \frac{1}{2}\,\mathrm{div}\,L) \tag{2.3.4}$$

is studied in [App. 2]. The following proposition comprises the results of such study.

Proposition (3.1)

Let $L = \xi(x)d/dx$ be a C^∞ vector field in M. Consider $\hat{P}_0$ given by (2.3.4) with domain

$$D_{P_0} = \{\psi | \psi \in C_0^1(M) ; \psi , \hat{P}\psi \in L^2(M)\} .$$

(i) $\hat{P}_0$ is symmetric .

(ii) The adjoint of P_0 is

$$\hat{P}_0^+ = -ih \ (L + \frac{1}{2} \operatorname{div} L)$$

with domain

$$D_{P_0^+} = \{\psi | \psi \in C^1(M) ; \psi , \hat{P}_0^+\psi \in L^2(M)\} .$$

(iii) If L is complete, then $\hat{P}_0$ is essentially self-adjoint with a unique self-adjoint extension $\hat{P}_0^+$.

APPENDIX 1

For definitions of the concepts of symmetry, self-adjointness, essential self-adjointness or adjoint of an operator we refer to [1-3]. Before going into the proofs of propositions (2.1) and (2.2) we prove two lemmas.

Lemma (A1.1). Let I be the identity operator in a Hilbert space H and let A be a symmetric operator in $D_A \subseteq H$ whose adjoint is A^+. A is essentially self-adjoint iff

$$(A^+ \pm iI)\psi = 0 \ , \ \psi \in D_{A^+} \Rightarrow \psi = 0 . \tag{a1.1}$$

Proof: By definition A is essentially self-adjoint if $(A \pm iI)D_A$ are dense in H. Now consider

$$(\psi, (A \pm i)\phi) = ((A^+ \mp i)\psi, \phi),$$

where $\psi \in D_{A^+}$, $\phi \in D_A$.

$$(A^+ \mp i)\psi = 0 \Rightarrow (\psi, (A \pm i)\phi) = 0 \quad \forall \phi \in D_A .$$

Thus $\psi = 0$ if A is essentially self-adjoint.

Conversely, suppose (a1.1) is true and let $\psi \in H$ such that

$$((A \pm i)\phi, \psi) = 0 \quad \forall \phi \in D_A .$$

This above relation could be written in the form

$$((A \pm i)\phi, \psi) = (\phi, 0) \quad \forall \phi \in D_A .$$

By the definition of an adjoint of an operator we have

$$\psi \in D_{A^+} \quad \text{and} \quad (A^+ \mp i)\psi = 0 .$$

By (a1.1) we have $\psi = 0$, and hence A is essentially self-adjoint. ∎

Let S be the set of all functions ψ in $C_0^1(M)$ with a common support $[x_1,x_2]$.

Lemma (A1.2). If $f \in L^2(M)$ is such that $(f,\frac{d\psi}{dx}) = 0 \ \forall \psi \in S$, then $f = \text{constant}/\sqrt{g}$ in $[x_1,x_2]$.

Proof: By assumption we have

$$\int_{x_1}^{x_2} f^* \frac{d\psi}{dx} \sqrt{g}\, dx = 0 \quad \forall \psi \in S.$$

integrating by parts, we get

$$\int_{x_1}^{x_2} \frac{d}{dx}(f^*\sqrt{g})\, \psi\, dx = 0 \quad \forall \psi \in S,$$

and hence

$$\int_{x^1}^{x^2} \Big[\frac{1}{\sqrt{g}} \frac{d}{dx}(f^*\sqrt{g})\Big] \psi \sqrt{g}\, dx = 0 \quad \forall \psi \in S.$$

Since S is dense in $L^2([x_1,x_2])$, we have

$$\frac{1}{\sqrt{g}} \frac{d}{dx}(f^*\sqrt{g}) = 0 \ \text{in}\ [x_1,x_2] \Rightarrow f^* = c/\sqrt{g} \ \text{in}\ [x_1,x_2]. \ \blacksquare$$

Proof of proposition (2.1)

(i) $\hat{p}$ is symmetric.

Let the range of x be (a,b), where a and b are finite or infinite numbers, and let $\psi,\phi \in D_p$. Now

$$(\psi,\hat{P}\phi) = \int_a^b \psi^*\Big\{-i\Big(\frac{d}{dx} + \tfrac{1}{2}(\ln\sqrt{g})_{,x}\Big)\phi\Big\}\sqrt{g}\, dx$$
$$= -i\int_a^b \psi^* \frac{d\phi}{dx}\sqrt{g}\, dx + \int_a^b \Big\{\frac{i}{2}(\ln\sqrt{g})_{,x}\psi\Big\}^* \phi\sqrt{g}\, dx.$$

(we set $\hbar = 1$). On integrating the first integral by parts, we find

$$(\psi,\hat{P}\phi) = -i[\psi^*(b)\phi(b)\sqrt{g(b)} - \psi^*(a)\phi(a)\sqrt{g(a)}]$$
$$+ \int_a^b \{-i(d\psi/dx + \tfrac{1}{2}(\ln\sqrt{g})_{,x}\psi)\}^* \phi\sqrt{g}\, dx. \quad \text{(a1.2)}$$

Since $\psi,\phi \in D_p$, the integrated part vanishes yielding

$$(\psi,\hat{P}\phi) = (\hat{P}\psi,\phi) \quad \forall \psi,\phi \in D_p \quad \text{(a1.3)}$$

Moreover, the set of infinitely differentiable functions with compact support on M is dense in $L^2(M)$ [3]. But such a set is contained in D_p, and hence D_p is dense in $L^2(M)$. Thus $\hat{p}$ is symmetric.

(ii) The adjoint of $\hat{p}$.

Let $\phi \in D_p$ and $\psi \in D_{p^+}$, and examine the relation

$$(\psi, \hat{P}\phi) = (\xi, \phi). \tag{a1.4}$$

The left-hand side of this above relation may be written in the form

$$\begin{aligned}(\psi, \hat{P}\phi) &= \int_{x_1}^{x_2} \psi^* \{-i[d/dx + \tfrac{1}{2}(\ln\sqrt{g})_{,x}]\,\phi\}\sqrt{g}\,dx \\ &\qquad ([x_1, x_2] \text{ is the support of } \phi) \\ &= \int_{x_1}^{x_2} (i\sqrt{g}\,\psi)^* \frac{d\phi}{dx}\,dx + \int_{x_1}^{x_2} (\tfrac{i}{2}(\sqrt{g}\,)_{,x}\psi)^*\phi\,dx \\ &= \int_{x_1}^{x_2} (i\sqrt{g}\,\psi)^* \frac{d\phi}{dx}\,dx - \int_{x_1}^{x_2} \Big\{\int_{x_1}^{x} [(\tfrac{i}{2}\sqrt{g})_{,x}\psi]\,dx + c\Big\}^* \frac{d\phi}{dx}\,dx \\ &= \int_{x_1}^{x_2} \Big\{ i\sqrt{g}\,\psi - \Big[\int_{x_1}^{x} \tfrac{i}{2}(\sqrt{g})_{,x}\psi\,dx + c\Big]\Big\}^* \frac{d\phi}{dx}\,dx. \end{aligned} \tag{a1.5}$$

The right-hand side of (a1.4) may be written as

$$(\xi, \phi) = \int_{x_1}^{x_2} \xi^* \phi \sqrt{g}\,dx = -\int_{x_1}^{x_2} \Big[\int_{x_1}^{x} \xi\sqrt{g}\,dx + c_1\Big]^* \frac{d\phi}{dx}\,dx. \tag{a1.6}$$

Equating the right-hand sides of (a1.5) and (a1.6), we find

$$\int_{x_1}^{x_2} \Big\{ i\sqrt{g}\,\psi + \Big[\int_{x_1}^{x} (\xi\sqrt{g} - \tfrac{i}{2}(\sqrt{g})_{,x}\psi)\,dx + c_2\Big]^* \Big\}^* \frac{d\phi}{dx}\,dx = 0$$

By lemma (A1.2) the above equation gives

$$i\sqrt{g}\,\psi + \int_{x_1}^{x} (\xi\sqrt{g} - \tfrac{i}{2}(\sqrt{g})_{,x}\psi)\,dx + c_3 = 0 \quad \forall x \in [x_1, x_2]. \tag{a1.7}$$

This last relation holds for all such intervals $[x_1, x_2]$ in M, so it holds all over M. Also, (a1.7) shows that

ψ is a $C^1(M)$. Differentiating (a1.7) with respect to x and solving for ξ, we get

$$\xi = -i\,(d/dx + \tfrac{1}{2}(\ln\sqrt{g})_{,x})\psi ,$$

which completes the proof.

(iii) $\hat{p}$ is not self-adjoint. This is obvious since $\hat{p} \neq \hat{p}^{+}$. ∎

Proof of proposition (2.2)

1. The range of x is (o,b).

(i) $\hat{p}$ is not essentially self-adjoint.

Substituting for $\hat{p}^{+}$ from (2.2.3) in (A1.1), we find

$$-i\hbar\,[d/dx + \tfrac{1}{2}(\ln\sqrt{g})_{,x}]\psi = \pm i\,\psi \qquad \text{(a1.8)}$$

This equation admits two solutions ψ_+ and ψ_-, where

$$\psi_{\mp} = \frac{c_{\mp}}{\sqrt[4]{g}}\, e^{\pm x/\hbar} \quad (c_+ \text{ and } c_- \text{ are constants}).$$

Both ψ_+ and ψ_- are C^1 functions on M and both are in $L^2(M)$ since

$$\|\psi_{\mp}\|^2 = |c_{\mp}|^2 \int_0^b e^{\pm 2x/\hbar}\, dx < \infty .$$

It follows that $\pm i$ are eigenvalues of p^{+} in $D_{p^{+}}$. By lemma (A1.1) we deduce that p is not essentially self-adjoint.

(ii) Self-adjoint extensions of p.

If $\hat{P}$ is a proper symmetric extension of $\hat{p}$ with a domain $D_p \subset C^1(M) \subset L^2(M)$, then $\hat{P} \subseteq \hat{p}^{+}$ [1,2]. Thus

$$(\hat{P}\psi, \phi) = (\psi, \hat{P}\phi) = -i\hbar\,[\psi^*(b)\phi(b)\sqrt{g(b)} - \psi^*(0)\phi(0)\sqrt{g(0)}] + (\hat{P}\psi, \phi) \quad \forall\, \psi, \phi \in D_P .$$

It follows that

$$\psi^*(b)\,\phi(b)\sqrt{g(b)} = \psi^*(0)\,\phi(0)\sqrt{g(0)} \quad \forall\, \psi, \phi \in D_P . \qquad \text{(a1.9)}$$

Setting $\psi = \phi$ in this above relation, we find

$$|\phi(b)|^2\sqrt{g(b)} = |\phi(0)|^2\sqrt{g(0)}\,,$$

and hence

$$e^{i\beta}\,\phi(b)\sqrt[4]{g(b)} = \phi(0)\sqrt[4]{g(0)}\,, \qquad \text{(a1.10)}$$

where $\beta \in [0,2\pi)$. The condition (a1.10) is necessary for $\hat{P}$ to be self-adjoint since self-adjointness implies symmetry. Also, it is sufficient; for if $\hat{P}^+$ is the adjoint of the symmetric operator $\hat{P}$, then $\hat{P}^+ \subseteq \hat{p}^+$ and

$$(\psi, \hat{P}\phi) = -i\hbar\left[\psi^*(b)\phi(b)\sqrt{g(b)} - \psi^*(0)\phi(0)\sqrt{g(0)}\right] + \left(-i\hbar\left(d/dx + \tfrac{1}{2}(\ln\sqrt{g})_{,x}\right)\psi, \phi\right) \quad \forall\,\phi \in D_P\,.$$

By (a1.10),

$$(\psi, \hat{P}\phi) = -i\hbar\sqrt[4]{g(0)}\,\phi(0)\left[e^{-i\beta}\sqrt[4]{g(b)}\,\psi^*(b) - \psi^*(0)\sqrt[4]{g(0)}\right] + \left(-i\hbar\left[d/dx + \tfrac{1}{2}(\ln\sqrt{g})_{,x}\right]\psi, \phi\right) \quad \forall\,\phi \in D_P\,.$$

Now the requirement

$$(\psi, \hat{P}\phi) = (\hat{P}^+\psi, \phi) \quad \forall\,\phi \in D_P, \forall\,\psi \in D_{P^+}\,,$$

implies that

$$e^{i\beta}\,\psi(b)\sqrt[4]{g(b)} = \psi(0)\sqrt[4]{g(0)} \quad \forall\,\psi \in D_{P^+}\,.$$

Thus

$$\hat{P}^+ = -i\hbar\left(d/dx + \tfrac{1}{2}(\ln\sqrt{g})_{,x}\right) \qquad \text{(a1.11)}$$

with the domain

$$D_{P^+} = \left\{\psi \mid \psi \in C^1(M);\ \psi, \hat{P}\psi \in L^2(M);\ e^{i\beta}\psi(b)\sqrt[4]{g(b)} = \psi(0)\sqrt[4]{g(0)}\right\}.$$

Thus every self-adjoint extension $\hat{P}$ of $\hat{p}$ is given by (2.2.5) and (2.2.6).

(iii) Eigenfunctions and eigenvalues of $\hat{P}$.

On solving the eigenequation

$$-i\hbar\left(d/dx + \tfrac{1}{2}(\ln\sqrt{g})_{,x}\right)\psi = \lambda\,\psi\,, \qquad \text{(a1.12)}$$

we find

$$\psi(x) = \frac{c}{\sqrt[4]{g}}\,e^{i\lambda x/\hbar}\,. \qquad \text{(a1.13)}$$

Making use of the boundary conditions (a1.10), we find that $\hat{P}$ possesses the point spectrum

$$\lambda_n = (2n\pi - \beta)\hbar/b \quad (n = 0, \pm 1, \ldots) \qquad \text{(a1.14)}$$

Note: Strictly speaking, the points $x = 0$, $x = b$ are not in M and g may be zero at $x = 0$ or at $x = b$, i.e. $\psi(x)$ is not defined at these points. However, our argument may be replaced by a limiting process to obtain the same results.

2. The range of x is $(0,\infty)$.

(i) $\hat{p}$ is not essentially self-adjoint.

The solutions of the equation which result from the substitution for $\hat{p}^+$ from (2.3) in (a1.2) are

$$\psi_- = \frac{c_-}{\sqrt[4]{g}}\, e^{x/\hbar}, \quad \psi_+ = \frac{c_+}{\sqrt[4]{g}}\, e^{-x/\hbar}.$$

Now, $\psi_- \notin L^2(M)$ but ψ_+ does since

$$\|\psi_+\|^2 = |c_+|^2 \int_0^\infty e^{-2x/\hbar}\, dx = \frac{\hbar}{2}|c_+|^2 < \infty.$$

By lemma (A1.1), $\hat{p}$ is not essentially self-adjoint.

(ii) $\hat{p}$ has no self-adjoint extension.

Suppose that $\hat{P}$ is a self-adjoint extension of $\hat{p}$ whose domain $D_p \subset C^1(M) \subset L^2(M)$. If $\phi \in D_p$, and since $\hat{P}$ is symmetric, we have

$$(\phi, P\phi) = -i\hbar\left[|\phi(\infty)|^2\sqrt{g(\infty)} - |\phi(0)|^2\sqrt{g(0)}\right] + (\hat{P}\phi, \phi).$$

Since $\phi \in L^2(M)$, then $\lim_{x\to\infty} |\phi(x)|^2 \sqrt{g(x)} = 0$, and hence

$$|\phi(0)|^2\sqrt{g(0)} = 0 \quad \forall\, \phi \in D_P$$

$$\Rightarrow \quad \phi(0)\sqrt[4]{g(0)} = 0 \quad \forall\, \phi \in D_P. \qquad \text{(a1.15)}$$

Now consider the function $\phi_0 = e^{-x}/\sqrt[4]{g}$. This function is in $L^2(M)$ since

$$\|\phi_0\|^2 = \int_0^\infty e^{-2x}\, dx < \infty.$$

But

$$(\phi_0, \hat{P}\psi) = -i\hbar\, \phi_0^*(0)\, \psi(0)\sqrt{g(0)} + (-i\hbar(d/dx + \tfrac{1}{2}(\ln\sqrt{g})_{,x})\,\phi_0, \psi)$$
$$= -i\hbar\, \psi(0)\sqrt[4]{g(0)} + (-i\hbar(d/dx + \tfrac{1}{2}(\ln\sqrt{g})_{,x})\,\phi_0, \psi)$$
$$= (\hat{P}^+\phi_0, \psi).$$

Thus $\phi_o \in D_{p+}$, and hence $D_p \neq D_{p+}$. It follows that $\hat{P}$ is symmetric but not self-adjoint, i.e., $\hat{p}$ in this case has no self-adjoint extension.

3. The range of x is $(-\infty,\infty)$.

It is quite obvious in this case that p is essentially self-adjoint with a unique self-adjoint extension $\hat{p}^+$ (i.e., $\hat{p}^+ = \hat{p}^{++}$).

APPENDIX 2

A Study of the Operator $\hat{P}_o = -i(\xi d/dx + \frac{1}{2}\,\mathrm{div}(\xi d/dx))$ where $\xi d/dx$ is a C^∞ Vector Field in M; its Domain is

$$D_{P_o} = \{\psi \mid \psi \in C_0^1(M);\ \psi, P_o\psi \in L^2(M)\}$$

Proposition (A2.1).

(i) The adjoint of $\hat{P}_o$ is

$$\hat{P}_o^+ = -i(\xi\, d/dx + \frac{1}{2}\,div(\xi\, d/dx) \qquad \text{(a2.1)}$$

with domain

$$D_{P_o^+} = \{\psi \mid \psi \in C^1(M);\ \psi, \hat{P}_o^+\psi \in L^2(M)\} \qquad \text{(a2.2)}$$

(ii) $\hat{P}_o$ is symmetric.

(iii) $\hat{P}_o$ is not necessarily self-adjoint.

Proof: Let $\psi \in D_{P_o^+}$ and consider all the pairs $\phi, \xi \in L^2(M)$ such that

$$(\phi, \hat{P}_o\psi) = (\xi, \psi) \quad \forall \psi \in D_{P_o} \qquad \text{(a2.3)}$$

The left-hand side of (a2.3) may be written as follows

$$(\phi, P_o\psi) = \int_{x_1}^{x_2} \phi^*\{-i[\xi \frac{d}{dx} + \frac{(\xi\sqrt{g})_{,x}}{2\sqrt{g}}]\psi\}\sqrt{g}\,dx \quad ([x_1,x_2] \text{ is the support of } \psi)$$

$$= \int_{x_1}^{x_2} (i\xi\sqrt{g}\,\phi)^* \frac{d\psi}{dx}\,dx - \int_{x_1}^{x_2}\left\{\int_{x_1}^{x} \frac{i}{2}\,\phi(\xi\sqrt{g})_{,x} + c\right\}^* \frac{d\psi}{dx}\,dx$$

$$= \int_{x_1}^{x_2}\left\{i\xi\sqrt{g}\,\phi - \int_{x_1}^{x}\frac{i}{2}\,\phi(\xi\sqrt{g})_{,x}\,dx + c\right\}^* \frac{d\psi}{dx}\,dx\,. \qquad \text{(a2.4)}$$

The right-hand side of (a2.3) may be written as

$$(\xi, \psi) = \int_{x_1}^{x_2} \xi^*\psi\sqrt{g}\,dx = -\int_{x_1}^{x_2}\left\{\int_{x_1}^{x}\xi\sqrt{g}\,dx + c_1\right\}^* \frac{d\psi}{dx}\,dx\,. \qquad \text{(a2.5)}$$

From (a2.3), (a2.4) and (a2.5), we have

$$\int_{x_1}^{x_2} \left\{ i\xi\sqrt{g}\,\phi + \int_{x_1}^{x} [\xi\sqrt{g} - \frac{i}{2}\phi(\xi\sqrt{g})_{,x}]\,dx + c_2 \right\} \frac{d\psi}{dx}\,dx = 0.$$

By lemma (A1.2) we have

$$i\xi\sqrt{g}\,\phi + \int_{x_1}^{x} [\xi - \frac{i\phi(\xi\sqrt{g})_{,x}}{2\sqrt{g}}]\sqrt{g}\,dx + c_3 = 0. \qquad \text{(a2.6)}$$

This last relation shows that

$$\phi \in C^1(M) \text{ and } \xi = -i(\xi\frac{d}{dx} + \frac{(\xi\sqrt{g})_{,x}}{2\sqrt{g}})\phi, \qquad \text{(a2.7)}$$

and hence (i) is proved. It is clear that

$$(\phi, \hat{P}_0\psi) = (\hat{P}_0\phi, \psi) \quad \forall\, \psi, \phi \in D_{\hat{P}_0}.$$

Thus $\hat{P}_o$ is symmetric (it is clear that $\bar{D}_{P_o} = L^2(M)$). Since $\hat{P}_o \neq \hat{P}_o^+$, $\hat{P}_o$ is not self-adjoint. $\hat{P}_o$ is not in general essentially self-adjoint. This is most easily seen by referring to the special case in which $L = \frac{d}{dx}$, $\sqrt{g} = 1$ and $M = (0,\infty)$ or (a,b). ∎

Proposition (A2.2).

If the C^∞ vector field $L = \xi d/dx$ is complete in M, then $\hat{P}_o$ is essentially self-adjoint.

For a proof see (Wan and Viazminsky [26]). ∎

Proposition (A2.3).

If the C^∞ vector field $L = \xi d/dx$ is complete in M, then the restriction $\hat{P}$ of $\hat{P}_o$ to $C_o^\infty(M)$ is again essentially self-adjoint.

The proof lies in the demonstration that $\hat{P}^+ = \hat{P}_o^+$ which is easy to achieve. ∎

CHAPTER III

Quantization in Spaces of Constant Curvature

§1. Introduction

Having found the canonical quantization scheme unsatisfactory, we seek a quantization procedure which does not make use of canonical variables. Better still, the scheme should be coordinate independent and hence applicable to a general Riemannian manifold. An approach formulated by Mackey [4] meets our requirements. This approach is summarised in the next section.

Though it is powerful and rigorous, Mackey's scheme seems not to fulfil some physically desirable requirements. Classically, when a particle displays a free motion in some manifold, nature provides some quantities which remain constant throughout the motion. Obviously, very few physical quantities fulfil this property. Those which do are of outstanding physical importance since they are the constituents of the conservation laws which are the corner stones of physics. We would like any mathematical scheme describing the physical world to single out those conserved quantities as being of genuine physical meaning.

The property of self-adjointness is a first requirement for a quantum momentum operator, and Mackey's scheme provides always such a property. However, the momenta produced by the scheme are not necessarily conserved unless an extra condition is imposed on them. In §3 we will give some examples to illustrate some features of

Mackey's scheme when it is applied to some Riemannian manifolds. Mackey's momenta which are conserved are obtained by requiring that the the corresponding vector fields are Killing fields, and in order to accord with self-adjointness of the momenta, these Killing vector fields are required to be complete.

Obviously, conserved momenta in a general Riemannian manifold may not exist at all. If that happened to be the case, then we cannot impose any extra condition on Mackey's momenta, and we do not have an alternative or a modification of his scheme.

Spaces of constant curvature possess enough symmetry so that our modification can play its role. The problem of quantization in such spaces is discussed in §4-§8. Explicit forms of the quantum and classical momenta are found. An interesting relation in any N-dimensional space of constant curvature between the Hamiltonian, the momenta and the curvature of the space holds in the classical and quantal cases. In §9 we will show that the quantum and classical momenta could be made into isomorphic Lie algebras.

§2. Mackey's Quantization Scheme

Suppose that the configuration space of a physical system is a C^{∞} Riemannian manifold M with metric g_{mn}. In classical mechanics the phase space is then the cotangent bundle T*M. If x^i is any chosen system of coordinates in M, then T*M may be coordinatized by $\{x^i, p_i\}$, p_i being the generalized momenta. Let U be an OPG of transformations (diffeomorphisms) of M. The infinitesimal generator $L = \xi^i \partial/\partial x^i$ of U is a complete vector field in M [27,28]. U induces an OPG Φ of T*M whose generator ϕ is a globally Hamiltonian vector field on T*M; that is, there is a function P on T*M such that ϕ is the contravariant counter-part of the covariant vector field dP. Generally a classical observable is a function on T*M and any function on T*M which is related to an OPG of M in this fashion is referred to as a momentum (not to be confused with a generalized momentum).

Furthermore, every OPG of M may be shown to define an OPG V_t of unitary transformations of $L^2(M)$. By Stone's theorem [25,p335], there corresponds to V_t a self-adjoint operator $\hat{P}$ such that $V_t = e^{i\hat{P}t}$. Hence, for each OPG of M there corresponds on the one hand a classical momentum observable P on the phase space T*M and on the other hand a self-adjoint operator $\hat{P}$ on the Hilbert space $L^2(M)$. It is therefore natural to postulate that on quantization P goes to $\hat{P}$. This is epistemologically pleasing. Indeed it is just an extension of the well-known situation when one quantizes classical momentum in Cartesian coordinates in Euclidean space. The procedure depends on the geometric properties of M and is coordinate independent.

Let $C_0^\infty(M)$ denote the set of C^∞ functions of compact support on M. Then the restriction of $\hat{P}$ on $C_0^\infty(M)$ takes the form of a differential operator

$$\hat{P}\psi = -i\hbar\,(L + \tfrac{1}{2}\,\mathrm{div}\,L)\,\psi\,,\ \psi \in C_0^\infty(M)\,. \tag{3.2.1}$$

This restriction of $\hat{P}$ is essentially self-adjoint. In other words, $\hat{P}$ is the unique self-adjoint extension of the above differential operator. Appendix 2 in the last chapter demonstrates that

$$\hat{P} = -i\hbar\,(L + \tfrac{1}{2}\,\mathrm{div}\,L) \tag{3.2.2}$$

with the domain

$$D_P = \{\psi \mid \psi \in C^1(M);\ \psi, \hat{P}\psi \in L^2(M)\}\,. \tag{3.2.3}$$

In Mackey's scheme the Hamiltonian is assumed to be proportional to the Laplacian. Mackey's scheme also deals with the problem of coordinate variables and their functions. The result is that associated with every real-valued Borel function f defined everywhere on the manifold M there corresponds a quantum observable which is just the familiar self-adjoint operation of multiplication by f in the Hilbert space $L^2(M)$. A coordinate variable x^i, if defined throughout M, is a Borel function on M. Hence, it may be quantized in the usual way. The situation is different if x^i is not defined throughout M as would be the case when the manifold M cannot be covered by a single chart.

We are now in a position to re-examine the canonical quantization scheme to see why it sometimes breaks down. Let us consider a manifold M which is coverable by a single coordinate chart x^i. Classically, a generalized momentum p_i may be regarded as a function on T*M. As such, if it is associated with an OPG of M in the manner described above, p_i may be satisfactorily quantized to give a self-adjoint quantum momentum observable $\hat{P}_i$ whose restriction to $C_0^\infty(M)$ is

$$-i\hbar(\partial/\partial x^i + \frac{1}{2}\,\mathrm{div}(\partial/\partial x^i)), \tag{3.2.4}$$

where $\partial/\partial x^i$ is the generator of the OPG. In this case we are justified in calling $\hat{x}^i, \hat{p}_i$ canonical variables. The crucial criterion is that p_i must generate an OPG of M. Otherwise the differential operator (3.2.4) will not be essentially self-adjoint. Consequently, we will not have a well-defined quantum momentum observable. The fact that the generator is the vector field $\partial/\partial x^i$ tells us that the OPG generated by the generalized momentum associated with a generalized coordinate x^i is the group of translation of the coordinate x^i. However, translations form a group only if x^i is allowed the range $(-\infty, +\infty)$. This is precisely why a quantum canonical momentum observable can be established only for a coordinate variable which takes the range $(-\infty, +\infty)$.

§3. Some Applications of Mackey's Scheme

§(3.1) One-Dimensional Manifolds

Throughout this subsection and unless indicated to the contrary we will assume that the metric form of M is $ds^2 = dx^2$. Also we note that one way of finding an OPG of M is to find a diffeomorphism $F:M\to\mathbb{R}$. If F is so, then $U_t(M) = F^{-1}(t + F(m))$ $(t\in\mathbb{R}, m\in M)$ is an OPG of M . The existence of such a diffeomorphism implies that M is coverable by a single chart, namely $m\to F(m)$. If M is not, then we have to appeal to another way in order to find an OPG of M.

§(3.1.1) M = the real line $\mathbb{R}$

An OPG of $\mathbb{R}$ is the group of translations $U_t(x) = x + t$. The infinitesimal generator of U_t is $L = d/dx$. Since $\operatorname{div} L = 0$, the momentum operator takes the form $\hat{P} = -i\hbar d/dx$. Because L is complete, the operator $\hat{P}$ with a domain $D_p = \{\psi \mid \psi \in C^1(M); \psi, \hat{P}\psi \in L^2(M)\}$ is self-adjoint. The Hamiltonian operator of a free particle is $-\frac{\hbar^2}{2}\nabla^2 = \frac{1}{2}\hat{P}^2$. The domain of $\hat{H}$ is $D_H = \{\Psi \mid \Psi \in D_p, \hat{P}\Psi \in D_p\}$ [2.p318]. The Hamiltonian with the previous domain is self-adjoint.

Let y be a coordinate system on $\mathbb{R}$ in terms of which the metric assumes the form

$$ds^2 = g\, dy^2. \tag{3.3.1}$$

Let $L_y = \xi\, d/dy$ be the infinitesimal generator of an OPG of $\mathbb{R}$. If we require L_y to be volume-preserving [28], then

$$\operatorname{div} L = \frac{1}{\sqrt{g}} (\sqrt{g}\ \xi)_{,y} = 0 . \tag{3.3.2}$$

The solution of the above differential equation is

$$\xi = k/\sqrt{g}\ , \tag{3.3.3}$$

where k is an arbitrary constant. Taking k = 1, the momentum operator assumes the form

$$\hat{P}_y = (-i\hbar/\sqrt{g})\, d/dy \,. \tag{3.3.4}$$

From (3.3.4) the quantum Hamiltonian is

$$-\frac{\hbar^2}{2}\nabla^2 = -\frac{\hbar^2}{2}\frac{1}{\sqrt{g}}\cdot\frac{d}{dy}\left(\frac{1}{\sqrt{g}}\frac{d}{dy}\right) = \frac{1}{2}\hat{P}_y^{\,2}.$$

If we let for example , $y = \arctan x$, then

$$ds^2 = dx^2 = dy^2/\cos^4 y.$$

The OPG $U_t(y) = \arctan(t + \tan y)$ has the infinitesimal generator $L_y = \cos^2 y\; d/dy$ which is volume-preserving. The momentum operator $\hat{P}_y$ and the Hamiltonian $\hat{H}$ are given by

$$\hat{P}_y = -i\hbar \cos^2 y\; d/dy \quad , \quad \hat{H} = -\tfrac{1}{2}\hbar^2(\cos^2 y\; d/dy)^2.$$

In terms of x, the OPG is just the translation group $U_t = x + t$.

We may perform on $\mathbb{R}$ another OPG $U_t'(x) = xe^t$. The generator $L = x\; d/dx$ of U_t' is of divergence equal to 1. The corresponding momentum operator is $\hat{P} = -i\hbar(x\; d/dx + \tfrac{1}{2})$. $\hat{P}$ is self-adjoint when the usual domain (3.2.3) is chosen. The free Hamiltonian is $-\tfrac{1}{2}\hbar^2\nabla^2$ as usual, but it is no longer proportional to the square of the momentum operator. We note that the classical momentum $P = xp$ arising from U_t' is not a constant of the free motion and possesses no obvious physical significance.

§(3.1.2) M = the semi-real line $(0, \infty)$

While the translation of the coordinate is not an OPG of M, the transformations $U_t(x) = xe^t$ form an OPG of M. The generator L of U_t, the momentum operator and the Hamiltonian are $x\; d/dx$, $-i\hbar(x\; d/dx + \tfrac{1}{2})$ and $-\tfrac{1}{2}\hbar^2 \dfrac{d^2}{dx^2}$ respectively. The momentum operator with the domain (3.2.3) is self-adjoint but is not compatible with the Hamiltonian.

§(3.1.2) M = the open interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$

Since $F(x) = \tan x$ is a diffeomorphism from M to $\mathbb{R}$, we have an OPG of M given by

$$U_t(x) = \arctan(t + \tan x). \qquad (3.3.5)$$

The infinitesimal generator of U_t is $L = \cos^2 x \, d/dx$. The corresponding quantum momentum operator is given by

$$\hat{P} = -i\hbar \cos^2 x \, (d/dx - \tan x).$$

Another OPG of M is $U_t'(x) = \mathrm{th}(t + \mathrm{Argth}(\frac{2}{\pi} x))$. U_t leads to the momentum operator

$$\hat{P}' = \frac{-i\hbar}{1-(\frac{2}{\pi}x)^2} \left\{ \frac{d}{dx} - \frac{4}{\pi^2} \frac{x}{1-(\frac{2}{\pi}x)^2} \right\}.$$

Both $\hat{P}$ and $\hat{P}'$, when their domains are given by (3.2.3) are self-adjoint. It follows that both $\hat{P}$ and $\hat{P}'$ are qualified as momentum operators. Niether U_t nor U_t' preserves the volume-element. The classical momenta $P = \cos^2 x \, p$ and $P' = p/(1 - (\frac{2}{\pi} x)^2)$, arising from U_t and U_t' respectively, are not constants of the free motion. The same is true for the quantum momenta.

§(3.1.4) M = the circle S^1

S^1 is a compact manifold, and hence every C^∞ vector field in S^1 is complete. However, the only volume-element-preserving OPG of S^1 is that which is generated by the vector field $d/d\phi$ described in chapter II.

§(3.2) Regular Surfaces in E^3

§(3.2.1) Let E^3 be the 3-dimensional Euclidean space and (r,ϕ,z) be a cylindrical coordinate system in E^3. Let M be a regular surface [36] $r = r(\phi,z)$ in E^3 and assume that ϕ takes all values in $[0,2\pi)$. The metric form of the 2-dimensional manifold M is

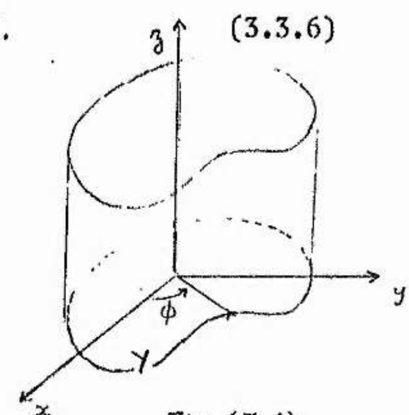

Fig.(3.1)

$$ds^2 = (r^2 + r_{,\phi}^2)\, d\phi^2 + 2\, r_{,z}\, r_{,\phi}\, d\phi\, dz + (1 + r_{,z}^2)\, dz^2. \qquad (3.3.6)$$

If z can vary from $-\infty$ to $+\infty$, then

$$U_t^1(\phi,z) = (\phi+t,z) \;,\quad U_t^2(\phi,z) = (\phi,z+t)$$

are OPG's of M which yield the momenta

$$\hat{P}_1 = -i\hbar\,(\partial/\partial\phi + \tfrac{1}{2}\,div(\partial/\partial\phi)),$$

$$P_2 = -i\hbar(\partial/\partial z + \tfrac{1}{2}\,div(\partial/\partial z)),$$

respectively.

§(3.2.2) If r is independent of z, then the metric form (3.3.6) is reduced to

$$ds^2 = (r^2 + r_{,\phi}^2)\, d\phi^2 + dz^2. \qquad (3.3.7)$$

Introducing instead of ϕ a new coordinate Y by

$$Y = \int_0^{\phi} (r^2 + r_{,\phi}^2)^{\frac{1}{2}}\, d\phi,$$

the metric form (3.3.7) becomes

$$ds^2 = dY^2 + dz^2 . \qquad (3.3.9)$$

This form shows that M is a developable surface (its Gaussian curvature $K = 0$ [36]). Because of (3.3.9) we may say that M is a flat space. In fact, M is a cylinder, with $r = r(\phi,0)$ as a base, and a generator parallel to oz (Fig.(3.1)). The OPG's $U_t^1(Y,z) = (Y+t,z)$ and $U_t^2(Y,z) = (Y,z+t)$ are motions of M. The corresponding momenta are

$$\hat{P}_1 = -ih\, \partial/\partial Y \;,\quad \hat{P}_2 = -ih\, \partial/\partial z \;. \qquad (3.3.10)$$

Contrary to the case encountered in §(3.2.1), the Hamiltonian $\hat{H}$ is closely related to the momentum operators:

$$\hat{H} = -\frac{\hbar^2}{2}\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial z^2}\right) = \frac{1}{2}\,(\hat{P}_1^2 + \hat{P}_2^2)\,. \qquad (3.3.11)$$

If Y is proportional to ϕ, then r is independent of ϕ as well and so it is a constant. The surface in this case is a circular cylinder.

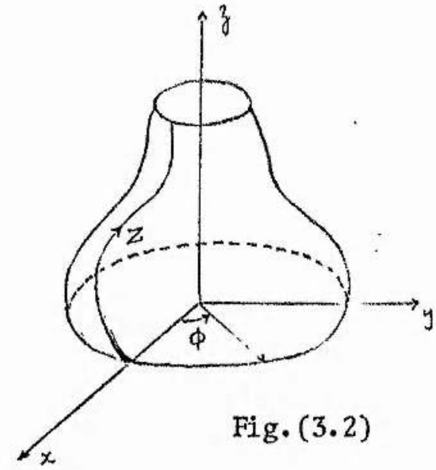

Fig.(3.2)

§(3.2.3) If r is independent of ϕ, then M is a surface of revolution. The metric form (3.3.6) becomes

$$ds^2 = r^2(z)\,d\phi^2 + (1+r_{,z}^2)\,dz^2. \qquad (3.3.12)$$

Introducing instead of z a new coordinate $Z = \int_0^z (1+r_{,z}^2)^{\frac{1}{2}}dz$, we may write (3.3.12) in the form

$$ds^2 = \rho^2(Z)\,d\phi^2 + dZ^2, \qquad (3.3.13)$$

where $\rho(z) = r(z(Z))$. The group $U_t^1(\phi,Z) = (\phi+t, Z)$ is a motion of M. The momentum $\hat{P} = -ih\,\partial/\partial\phi$ arising from it commutes with the Hamiltonian. If Z can vary from $-\infty$ to $+\infty$, then $U_t^2(\phi,Z) = (\phi,Z+t)$ is an OPG of M. However it is not a motion of M in general. The momentum operator $\hat{P}_Z = -ih(\partial/\partial Z + \rho_{,Z}/2\rho)$ arising from U^2 does not commute with the Hamiltonian.

A surface of revolution is not developable in general (is not a flat space). The relation (3.3.13) shows that if $\rho(Z)$ is constant, then the surface is developable; it is a circular cylinder. Also, if $\rho(Z) = \alpha Z$, then the surface is developable; it is a circular cone. However, we have to take only one branch of the cone and cut out its vertex so that the surface is a connected differentiable manifold. The only motion of the cone is $(\phi+t, Z)$. This motion yields a momentum operator $-ih\partial/\partial\phi$ which is compatible with $\hat{H}$. Translations along oZ do not form a group. The OPG (ϕ, Ze^t), is not a motion. The momentum operator $-ih(Z\,\partial/\partial Z + 1)$ arising from this group is not compatible with $\hat{H}$.

From the above examples we anticipate that if a space possesses no symmetry whatsoever (we use the word symmetry as a synonym of motion), then it is not possible to find a quantum momentum which is compatible with H. It can easily be proved that a vector field $L = \xi^j \partial/\partial x^j$ does not commute with the Laplacian unless it is a Killing vector. Moreover, a Killing vector is volume-preserving.* It follows that a quantum momentum $-i\hbar(L + \frac{1}{2}\mathrm{div}\, L)$ is compatible with $\hat{H}$ iff L is a Killing vector. Thus, unless the space possesses some symmetry, a conserved momentum never exists.

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* $$\mathrm{div}\, L = (\sqrt{g}\, \xi^j)_{,j}/\sqrt{g} = (1/2g) g_{,j}\, \xi^j + \xi^j_{,j}$$

$$= \tfrac{1}{2} g^{mn} g_{mn,j}\, \xi^j + \xi^j_{,j} \,.\quad [38,\text{p}123].$$

By (1.2.14) we have

$$\mathrm{div}\, L = -\tfrac{1}{2} g^{mn}(g_{mj}\, \xi^j_{,n} + g_{nj}\, \xi^j_{,m}) + \xi^j_{,j}$$

$$= -\tfrac{1}{2} \xi^j_{,j} - \tfrac{1}{2} \xi^j_{,j} + \xi^j_{,j} = 0.$$

§4. Construction of the Hamiltonian in Spaces of Constant Curvature

We shall consider complete Riemannian manifolds, that is, every geodesic is infinitely extendible in both directions [32]. This is physically important since in such a manifold a particle can execute free motion along a geodesic indefinitely [32]. Furthermore we shall confine ourselves to the study of an N-dimensional Riemannian manifold M of constant curvature. The curvature may be positive or negative, the sphere S^2 and the hyperbolic plane being non-trivial 2-dimensional examples. There are $N(N+1)/2$ independent complete Killing vector fields $L_\mu = \xi^i_\mu\, \partial/\partial x^i$ ($\mu=1, \ldots, N(N+1)/2$) in M. These vector fields constitute a basis of a real vector space of Killing vector fields. Every Killing vector generates an OPG of M [34], i.e. it is complete. The corresponding classical momenta on T*M are $P_\mu = \xi^i_\mu\, p_i$. These dynamical variables may be satisfactorily quantized according to (3.2.1) leading to the operators $\hat{P}_\mu = -ih\, \xi^i_\mu\, \partial/\partial x^i$.

§(4.1) The Idea

Our thesis is that for a free particle the quantum Hamiltonian is determined by the quantum momentum observables $\hat{P}_\mu$ up to a multiplicative constant and an additive constant. The argument runs as follows. Classically we know that each momentum P_μ is a constant of free motion. In quantum theory we make the postulate that the corresponding quantum momentum observables $\hat{P}_\mu = -ih\, \xi^i_\mu\, \partial/\partial x^i$ are also constants of free motion. Consequently the Hamiltonian $\hat{H}$ must be such that

$$[\hat{H}\, ,\, -ih\, \xi^i_\mu\, \partial/\partial x^i]\, f = 0 \quad , \forall \mu, \qquad (3.4.1)$$

where $f \in C^\infty_0(M)$. It will be shown in the next subsection that these equations serve to determine H explicitly provided we assume that H is a differentiable operator of the second order.

§(4.2) Explicit Construction of H

In an N-dimensional space of constant curvature K (a CC^N for short)* there exists a system of coordinates $(x^1, \ldots, x^N)$ such that the metric takes the form

$$ds^2 = dx^i dx^i / G, \tag{3.4.2a}$$

where

$$G = \left(1 + \frac{K}{4} x^i x^i\right)^2. \tag{3.4.2b}$$

This metric is known as the Riemannian form of a CC^N [34].

Assuming N =3, the Killing equations may be written in the form

$$\xi^i_{,i} = G_{,\ell}\, \xi^\ell / 2G \qquad (i=1,2,3;\ \text{no summation over } i), \tag{3.4.3a}$$

$$\xi^i_{,j} + \xi^j_{,i} = 0 \qquad (i,j=1,2,3; i \neq j). \tag{3.4.3b}$$

The differentiation of (3.4.3a) and (3.4.3b) with respect to x^i and x^j respectively gives

$$\xi^i_{,ij} = \xi^j_{,jj}\ ,\ \xi^i_{,ji} = -\xi^j_{,ii}\ , \tag{3.4.4}$$

(i,j=1,2,3; no summation over i or j). These last two sets of equations render

$$\xi^j_{,ii} = -\xi^j_{,jj} \quad (i \neq j; i,j=1,2,3;\ \text{no summation over } i \text{ or } j). \tag{3.4.5}$$

Now, we assume that the quantum Hamiltonian $\hat{H}$ should be a second-order differential operator in $L^2(M)$ of the form

$$\hat{H} = a^{mn}(\underline{x}) \frac{\partial}{\partial x^m} \frac{\partial}{\partial x^n} + b^m(\underline{x}) \frac{\partial}{\partial x^m} + c(\underline{x}), \tag{3.4.6}$$

where $a^{mn}(\underline{x})$, $b^m(\underline{x})$, and $c(\underline{x})$ are C^∞ functions of the spatial coordinates and $\det(a^{mn}) \neq 0$.

Proposition(4.1). The quantum Hamiltonian is uniquely determined by (3.4.1) and (3.4.6) as $\alpha\nabla^2 + \beta$, where α and β are arbitrary constants.

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* Spaces of constant curvature (CC^N) are denoted by CC^N_+, CC^N_-

Before going into the proof of this proposition we list a few lemmae whose proofs can be found in appendices 1,2 and 3.

Lemma (4.1). In the above system of coordinates a set of six independent Killing vector fields may be chosen to be

$$\begin{aligned}
L_1 &= \{\tfrac{K}{4}[(x^1)^2+(x^2)^2-(x^3)^2]+1\}\partial/\partial x^1+\tfrac{K}{2}x^1x^2\partial/\partial x^2+\tfrac{K}{2}x^1x^3\partial/\partial x^3,\\
L_2 &= \{\tfrac{K}{4}[(x^2)^2-(x^1)^2-(x^3)^2]+1\}\partial/\partial x^2+\tfrac{K}{2}x^2x^1\partial/\partial x^1+\tfrac{K}{2}x^2x^3\partial/\partial x^3,\\
L_3 &= \{\tfrac{K}{4}[(x^3)^2-(x^1)^2-(x^2)^2]+1\}\partial/\partial x^3+\tfrac{K}{2}x^3x^1\partial/\partial x^1+\tfrac{K}{2}x^3x^2\partial/\partial x^2,\\
L_4 &= x^2\partial/\partial x^3-x^3\partial/\partial x^2 \equiv L_{23},\\
L_5 &= x^3\partial/\partial x^1-x^1\partial/\partial x^3 \equiv L_{31},\\
L_6 &= x^1\partial/\partial x^2-x^2\partial/\partial x^1 \equiv L_{12},
\end{aligned}\tag{3.4.7}$$

Lemma (4.2). The matrix (ξ^i_μ), where $L_\mu = \xi^i_\mu\,\partial/\partial x^i$ $(i=1,2,3;$ $\mu=1,\ldots,6)$, is of rank 3 everywhere.

Lemma (4.3). If $b^m\,\partial/\partial x^m$ is a vector field in a CC^3 such that

$$[b^m\,\partial/\partial x^m, L_\mu] = 0 \qquad \forall\mu,$$

then $b^m\,\partial/\partial x^m = 0$.

Proof of proposition (4.1). By (3.4.6), the conditions (3.4.1) may be written in the form

$$(a^{mj}\xi^n_{\mu,j}+a^{nj}\xi^m_{\mu,j}-a^{mn}_{,j}\,\xi^j_\mu)\frac{\partial}{\partial x^m}\frac{\partial}{\partial x^n}+(a^{nj}\xi^m_{\mu,nj}+b^j\xi^m_{\mu,j}-b^m_{,j}\,\xi^j_\mu)\frac{\partial}{\partial x^m} - \xi^j_\mu c_{,j} = 0 \quad (\mu=1,\ldots,6;\, m,n,j=1,2,3). \tag{3.4.8}$$

This set of operator equations is satisfied iff the system of partial differential equations

$$\begin{aligned}
&a^{mj}\xi^n_{\mu,j}+a^{nj}\xi^m_{\mu,j}-a^{mn}_{,j}\,\xi^j_\mu = 0, && \mathrm{I}\\
&a^{nj}\xi^m_{\mu,nj}+b^j\xi^m_{\mu,j}-b^m_{,j}\,\xi^j_\mu = 0, && \mathrm{II}\\
&\xi^j_\mu c_{,j} = 0 && \mathrm{III}
\end{aligned}\tag{3.4.9}$$

is satisfied. If we substitute in I for a^{mn} by αg^{mn} (α is a constant and g^{mn} is the metric tensor), then the resultant equations are just

---

* or CC^N_0 according to K being positive, negative or zero.

the Killing equations (1.2.15). Thus the tensor αg^{mn} is a solution for I. The point now is to prove that this solution is unique. It is clear from (3.4.6) that a^{mn} is a tensor of the type (2,0). Thus a^{mn} may be taken as the contravariant components of the metric form of a 3-dimensional Riemannian space. But since six independent Killing vectors satisfy I, a^{mn} must be the metric of a space of constant curvature, and hence $a^{mn} = \alpha g^{mn}$. Thus the set I has a unique solution which may be written in terms of $(x^1, \ldots, x^N)$ as

$$a^{mj} = \alpha G\, \delta^{mj} \qquad (m, j = 1,2,3). \tag{3.4.10}$$

By (3.4.10) II may be written in the form

$$\alpha G\Big(\sum_{n=1}^{3} \xi^m_{\mu,nn}\Big) + b^j \xi^m_{\mu,j} - b^m_{,j}\, \xi^j = 0 \quad (m = 1,2,3;\ \mu = 1,\ldots,6) \tag{3.4.11}$$

which on account of (3.4.5) may be written as

$$-\alpha G\, \xi^m_{\mu,mm} + b^j \xi^m_{\mu,j} - b^m_{,j}\, \xi^j_\mu = 0 \qquad \text{(no summation over } m). \tag{3.4.12}$$

From (3.4.3a) we have

$$2G\, \xi^m_{\mu,m} = G_{,j}\, \xi^j_\mu \qquad \text{(no summation over } m),$$

and hence

$$2G_{,m} \xi^m_{\mu,m} + 2G\, \xi^m_{\mu,mm} = G_{,jm}\, \xi^j_\mu + G_{,j}\, \xi^j_{\mu,m} \quad \text{(no summaton over } m). \tag{3.4.13}$$

Substituting for $G\,\xi^m_{\mu,mm}$ from (3.4.13) in (3.4.12), we get

$$\begin{aligned} &2\alpha G_{,m}\, \xi^m_{\mu,m} - \alpha G_{,jm}\, \xi^j_\mu - \alpha G_{,j}\, \xi^j_{\mu,m} \\ &+ 2 b^j \xi^m_{\mu,j} - 2 b^m_{,j}\, \xi^j_\mu = 0 \qquad \text{(no summation over } m). \end{aligned} \tag{3.4.14}$$

Now, we observe that the above equation is satisfied if we insert $b^j = -\frac{1}{2}\alpha G_{,j}$, for

$$\begin{aligned} &2\alpha G_{,m}\, \xi^m_{\mu,m} - \alpha G_{,jm}\, \xi^j_\mu - \alpha G_{,j}\, \xi^j_{\mu,m} - \alpha G_{,j}\, \xi^m_{\mu,j} + \alpha G_{,jm}\, \xi^j_\mu \\ &= 2\alpha G_{,m}\, \xi^m_{\mu,m} - \alpha G_{,j}\, (\xi^j_{\mu,m} + \xi^m_{\mu,j}) \\ &= 2\alpha G_{,m}\, \xi^m_{\mu,m} - \alpha G_{,m}\, (\xi^m_{\mu,m} + \xi^m_{\mu,m}) \quad (\text{by } (3.4.3b)) \\ &= 0. \end{aligned}$$

Thus $b^j = -\frac{1}{2}G_{,j}$ is a solution for II. Now, we shall show that this solution is unique. Suppose that there exists another solution $b^j = \xi^j \neq -\frac{1}{2}\alpha G_{,j}$. Substituting for b^j in (3.4.11) by ξ^j and $-\frac{1}{2}\alpha G_{,j}$ respectively and substracting we get

$$\xi^j_\mu \eta^m_{,j} - \eta^j \xi^m_{\mu,j} = 0 \quad (m=1,2,3;\ \mu=1,\dots,6), \tag{3.4.15}$$

where $\eta^j = \xi^j + \frac{1}{2}\alpha G_{,j}$. But (3.4.15) is satisfied iff $[\eta^j \partial/\partial x^j, L_\mu] = 0$ ($\mu=1, \dots ,6$). By lemma (4,2), $\eta^j = 0$ (j=1,2,3), and hence $\xi^j = -\frac{1}{2}\alpha G_{,j}$ (j=1,2,3). It follows that $b^j = -\frac{1}{2}\alpha G_{,j}$ (j=1,2,3) is a unique solution for II.

In regard of the system III, it is obvious that the only solution of the homogeneous system $\xi^j_\mu c_{,j} = 0$ ($\mu=1, \dots ,6$) is the trivial solution $c_{,1} = c_{,2} = c_{,3} = 0$ since the matrix (ξ^j_μ) is of rank 3 everywhere.

From the above results we have

$$H = \alpha\left(G \frac{\partial}{\partial x^\ell}\frac{\partial}{\partial x^\ell} - \frac{1}{2}G_{,\ell}\frac{\partial}{\partial x^\ell}\right) + \beta, \tag{3.4.16}$$

where α and β are arbitrary constants. We may write this in the covariant form

$$H = \alpha \nabla^2 + \beta, \tag{3.4.17}$$

which does not depend on a particular choice of coordinates.

§(4.3) A Remark on the Classical Hamiltonian

The classical Hamiltonian also can be obtained straight from symmetry considerations. We assume the classical Hamiltonian, which is a scalar on the cotangent bundle, to be quadratic in the generalized momenta p_i, i.e.

$$H = a^{mn}(\underset{\sim}{x})p_m p_n + b^m(\underset{\sim}{x})p_m + c(\underset{\sim}{x}),$$

where a^{mn}, b^m and c are C^∞ functions and $\det(a^{mn}) \neq 0$. Since H is a scalar the functions a^{mn}, b^{m} and c are contravariant tensors of the second, first and zero order respectively. We assume further that the classical momenta are constants of the free motion, i.e.

$$\{H, \xi_\mu^j p_j\} = 0 \quad (\mu=1, \ldots ,6)$$

These last conditions lead to the system of differential equations

$$\begin{array}{ll} \mathrm{I}' & a^{mj}\xi^n_{\mu,j} + a^{nj}\xi^m_{\mu,j} - a^{mn}_{,j}\xi^j_\mu = 0 , \\ \mathrm{II}' & b^m_{,j}\xi^j_\mu - b^j\xi^m_{\mu,j} = 0 , \\ \mathrm{III}' & \xi^j_\mu c_{,j} = 0 , \end{array}$$

$(m,n,j=1,2,3;\mu=1, \ldots ,6)$. The systems I′ and III′ are identical to I and III encountered in the quantal case. Thus $a^{mn} = \alpha g^{mn}$ and $c=\beta$, where α and β are constants. By lemma (4.3), $b^1=b^2=b^3=0$. Hence

$$H = \alpha g^{mn} p_m p_n + \beta .$$

§5. Quantization in CC^2.

§(5.1) Preliminaries.

The metric form of a CC^2 in terms of a parallel geodesic coordinate system (x,y) is [36]

$$ds^2 = dx^2 + \mathrm{ch}^2\sqrt{-K}\,x\,dy^2 \qquad \text{if } K<0, \qquad (3.5.1a)$$

$$ds^2 = dx^2 + dy^2 \qquad \text{if } K=0, \qquad (3.5.1b)$$

$$ds^2 = dx^2 + \cos^2\sqrt{K}\,x\,dy^2 \qquad \text{if } K>0. \qquad (3.5.1c)$$

The coordinate curves $y=y_0$ are geodesics orthogonal to the geodesic $x=0$. The coordinate curves $x=x_0$ are not geodesics [36]. If we set $dx=0$ in (3.5.1), then the distance Δs along the curve $x=x_0$ between two points (x_0,y_0) and $(x_0,y_0+\Delta y_0)$ never vanishes if $K\leqslant 0$, but it does vanish for $x=\pm\pi/2\sqrt{K}$ if $K>0$. Thus all geodesics $y=y_0$ (y_0 is a parameter) do not meet if $K\leqslant 0$, But they do meet at a distance $s=\pi/2\sqrt{K}$ from the geodesic $x=0$ if $K>0$. It can be shown that all geodesics emanating from a point 0 in a CC^2 never meet if $K\leqslant 0$, but they meet again at a distance $s=\pi/\sqrt{K}$ from 0 if $K>0$ [35]. This latter situation is familiar in the case of a sphere, where all geodesics drawn from a pole 0 meet again at the opposite pole $0'$.

Quantization in any space requires the knowledge of the global properties of the space. A mere knowledge of the metric form of a CC^2 cannot specify the global properties of a CC^2. Therefore, we confine ourselves to simple cases in which we can consider a CC_+^2 as a sphere in E^3, a CC_0^2 as the Euclidean plane E^2, and a CC_-^2 as a pseudo-sphere in the three dimensional Minkowski space M^3. The ranges of the coordinates (x,y) are

$$-\infty < x,y < +\infty \quad \text{if } K \leqslant 0, \qquad (3.5.2a)$$

$$-\pi/2\sqrt{K} < x < \pi/2\sqrt{K}\ , \quad -\pi/\sqrt{K} < y < \pi/\sqrt{K} \quad \text{if } K > 0\ . \qquad (3.5.2b)$$

Obviously, more than one chart is needed to cover a CC_+^2. However, our treatment is coordinate independent, so that we can extend results obtained in one chart to another.

Finally we note that if we perform the coordinate transformations

$$\theta = \sqrt{eK}\, x \quad , \quad \phi = \sqrt{eK}\, y, \qquad (3.5.3)$$

where $eK = |K|$, then the metric forms of CC_-^2 and CC_+^2 can be written as

$$ds^2 = \frac{-1}{K}\,(d\theta^2 + ch^2\theta\ d\phi^2) \qquad \text{if } K < 0, \qquad (3.5.4a)$$

$$ds^2 = \frac{1}{K}\,(d\theta^2 + \cos^2\theta\ d\phi^2) \qquad \text{if } K > 0. \qquad (3.5.4b)$$

The ranges of θ and ϕ are

$$-\infty < \theta, \phi < +\infty \qquad \text{if } K < 0, \qquad (3.5.5a)$$

$$-\pi/2 < \theta < \pi/2\ , \quad -\pi < \phi < \pi \qquad \text{if } K > 0. \qquad (3.5.5b)$$

§(5.2) The Infinitesimal Motions of CC^2 in Geodesic Coordinates.

It may be shown that an infinitesimal motion is of the form

$$L = (\alpha\, ch\sqrt{-K}\, y - \frac{\beta}{\sqrt{-K}}\, sh\sqrt{-K}\, y)\,\partial/\partial x$$

$$+\{-th\sqrt{-K}\, x(\alpha\, sh\sqrt{-K}\, y - \frac{\beta}{\sqrt{-K}}\, ch\sqrt{-K}\, y) + \gamma\}\frac{\partial}{\partial y}\ . \qquad (3.5.6)$$

in CC_-^2,

$$L = (\alpha \cos\sqrt{K}\, y + \frac{\beta}{\sqrt{K}} \sin\sqrt{K}\, y)\,\partial/\partial x$$

$$+\{\tan\sqrt{K}\, x(\alpha \sin\sqrt{K}\, y - \frac{\beta}{\sqrt{K}} \cos\sqrt{K}\, y) + \gamma\}\,\partial/\partial y \qquad (3.5.7)$$

in CC_+^2 and

$$L = (\alpha - \beta y)\,\partial/\partial x + (\gamma + \beta x)\,\partial/\partial y \qquad (3.5.8)$$

in CC_0^2 [App 4]. Giving the triplet (α,β,γ) in each of the relations (3.5.6), (3.5.7) and (3.5.8) the values (1,0,0), (0,0,1) and (0,1,0), we get three independent infinitesimal motions for each type of CC^2. These infinitesimal motions are

Case I $K<0$

$$L_1 = ch\sqrt{-K}\,y\,\partial/\partial x - th\sqrt{-K}\,x\,sh\sqrt{-K}\,y\,\partial/\partial y, \tag{3.5.9a}$$

$$L_2 = \partial/\partial y \quad , \tag{3.5.9b}$$

$$L_3 = \frac{1}{\sqrt{-K}}(-sh\sqrt{-K}\,y\,\partial/\partial x + th\sqrt{-K}\,x\,ch\sqrt{-K}\cdot y\,\partial/\partial y); \tag{3.5.9c}$$

Case II $K>0$

$$\begin{aligned} L_1 &= \cos\sqrt{K}\,y\,\partial/\partial x + \tan\sqrt{K}\,x\,\sin\sqrt{K}\,y\,\partial/\partial y\,, \\ L_2 &= \partial/\partial y \quad , \\ L_3 &= \frac{1}{\sqrt{K}}(-\sin\sqrt{K}\,y\,\partial/\partial x + \tan\sqrt{K}\,x\,\cos\sqrt{K}\,y\,\partial/\partial y); \end{aligned} \tag{3.5.10}$$

Case III $K=0$

$$\begin{aligned} L_1 &= \partial/\partial x \quad , \\ L_2 &= \partial/\partial y \quad , \\ L_3 &= x\,\partial/\partial y - y\,\partial/\partial x\,. \end{aligned} \tag{3.5.11}$$

§(5.3) The Momentum and Hamiltonian Observables.

§(5.3.1) The Quantum Case.

Applying Mackey's scheme, the corresponding quantum momentum observables to these previous complete vector fields (3.5.9),(3.5.10) and (3.5.11) take the form

$$\hat{P}_\mu = -i\hbar\, L_\mu \quad (\mu=1,2,3) \tag{3.5.12}$$

in every case when operating on $C_0^\infty(CC^2)$. It can be verified that in every case we have

$$[\hat{P}_1,\hat{P}_2] = ihK\,\hat{P}_3\,,\ [\hat{P}_3,\hat{P}_1] = ih\,\hat{P}_2\,,\ [\hat{P}_2,\hat{P}_3] = ih\,\hat{P}_1, \tag{3.5.13}$$

and

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 = \frac{1}{2m}(\hat{P}_1^2 + \hat{P}_2^2 + K\,\hat{P}_3^2) \tag{3.5.14}$$

$$= \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \sqrt{-K}\, th\sqrt{-K}x \frac{\partial}{\partial x} + \frac{1}{ch^2\sqrt{-K}x}\frac{\partial^2}{\partial y^2}\right) \text{ if } K<0, \quad (3.5.15a)$$

$$= \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} - \sqrt{K}\tan\sqrt{K}x \frac{\partial}{\partial x} + \frac{1}{\cos^2\sqrt{K}x}\frac{\partial^2}{\partial y^2}\right) \text{ if } K>0, \quad (3.5.15b)$$

$$= \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \qquad \text{if } K=0. \quad (3.5.15c)$$

§(5.3.2) The Classical Case.

The classical momenta corresponding to the sets (3.5.9), (3.5.10) and (3.5.11) of infinitesimal motions of CC^2 are the following sets of functions on $T^*(CC^2)$:

Case I $K<0$

$$\begin{aligned} P_1 &= ch\sqrt{-K}y\, p_x - th\sqrt{-K}x\, sh\sqrt{-K}y\, p_y, \\ P_2 &= p_y, \\ P_3 &= \frac{1}{\sqrt{-K}}(-sh\sqrt{-K}y\, p_x + th\sqrt{-K}x\, ch\sqrt{-K}y\, p_y); \end{aligned} \quad (3.5.16)$$

Case II $K>0$

$$\begin{aligned} P_1 &= \cos\sqrt{K}y\, p_x + \tan\sqrt{K}x\, \sin\sqrt{K}y\, p_y, \\ P_2 &= p_y, \\ P_3 &= \frac{1}{\sqrt{K}}(-\sin\sqrt{K}y\, p_x + \tan\sqrt{K}x\, \cos\sqrt{K}y\, p_y); \end{aligned} \quad (3.5.17)$$

Case III $K=0$

$$\begin{aligned} P_1 &= p_x, \\ P_2 &= p_y, \\ P_3 &= x p_y - y p_x. \end{aligned} \quad (3.5.18)$$

It can be verified that in every case we have the following Poisson bracket relations:

$$\{P_1,P_2\} = K P_3, \quad \{P_3,P_1\} = P_2, \quad \{P_2,P_3\} = P_1 \quad (3.5.19)$$

and

$$H = \frac{1}{2m}(P_1^2 + P_2^2 + K P_3^2) \quad (3.5.20)$$

$$= \frac{1}{2m}\left(p_x^2 + \frac{p_y^2}{ch^2\sqrt{-K}\,y} \right) \qquad \text{if } K<0 \qquad (3.5.21a)$$

$$= \frac{1}{2m}\left(p_x^2 + \frac{p_y^2}{\cos^2\sqrt{K}\,y} \right) \qquad \text{if } K>0 \qquad (3.5.21b)$$

$$= \frac{1}{2m}\left(p_x^2 + p_y^2 \right) \qquad \text{if } K=0\,. \qquad (3.5.21c)$$

We can see that while the forms (3.5.14) and (3.5.20) of quantum and classical Hamiltonians are the same, there is no simple relationship between (3.5.15) and (3.5.21).

§(5.4) Various Considerations

§(5.4.1) Spectra of the Momenta and the Hamiltonian

In terms of the coordinate system (θ,ϕ) defined by (3.5.3), the infinitesimal motions (3.5.10) and (3.5.9) take the forms

$$K>0 \qquad \begin{aligned} L_1 &= \sqrt{K}\,(\cos\phi\, \partial/\partial\theta + \tan\theta \sin\phi\, \partial/\partial\phi)\,, \\ L_2 &= \sqrt{K}\,\partial/\partial\phi\,, \\ L_3 &= -\sin\phi\,\partial/\partial\theta + \tan\theta\cos\phi\,\partial/\partial\phi)\,; \end{aligned} \qquad (3.5.22)$$

$$K<0 \qquad \begin{aligned} L_1 &= \sqrt{-K}\,(ch\,\phi\,\partial/\partial\theta - th\,\theta\, sh\,\phi\,\partial/\partial\phi)\,, \\ L_2 &= \sqrt{-K}\,\partial/\partial\phi\,, \\ L_3 &= -sh\,\phi\,\partial/\partial\theta + th\,\theta\, ch\,\phi\,\partial/\partial\phi)\,. \end{aligned} \qquad (3.5.23)$$

If we define a new set of momentum observables by

$$\hat{P}_1' = (-i\hbar/\sqrt{eK})L_1\,;\; \hat{P}_2' = (-i\hbar/\sqrt{eK})L_2\,,\; \hat{P}_3' = -i\hbar L_3\,, \qquad (3.5.24)$$

then the following relations hold

$$\left\{ \begin{aligned} &[\hat{P}_i',\hat{P}_j'] = i\hbar\,\varepsilon_{ijk}\hat{P}_k' \quad (\,\varepsilon_{ijk} \text{ is the permutation symbol}), \\ &\hat{H} = (K/2m)(\hat{P}_1'^2+\hat{P}_2'^2+\hat{P}_3'^2)\,; \end{aligned} \right. \qquad K>0 \quad (3.5.25)$$

$$\left\{ \begin{aligned} &[\hat{P}_1',\hat{P}_2'] = -i\hbar\hat{P}_3'\,,\; [\hat{P}_2',\hat{P}_3'] = i\hbar\hat{P}_1'\,,\; [\hat{P}_3',\hat{P}_1'] = i\hbar\hat{P}_2' \\ &\hat{H} = (-K/2m)(\hat{P}_1'^2+\hat{P}_2'^2-\hat{P}_3'^2)\,. \end{aligned} \right. \qquad K<0 \quad (3.5.26)$$

Let V_1 and V_2 be two CC^2 of the same type with curvatures K_1 and K_2

respectively. If $\hat{H}_1$ and $\hat{H}_2$ are the Hamiltonians in V_1 and V_2 respectively, then

$$\hat{H}_1/K_1 = \hat{H}_2/K_2 \cdot \tag{3.5.27}$$

($\hat{H}_1$ and $\hat{H}_2$ acts on functions in $L^2(V_1)$ and $L^2(V_2)$ respectively. This previous formula implies that they have the same domain, a fact which is not obvious at this stage. In the next subsection we will prove that it is legitimate to write down the formula (3.5.27)).

The relations (3.5.25) are encountered when studying the angular momentum operators and the same algebraic method [24,39] may be applied here to deduce the eigenvalues of $\hat{P}_\mu$ and $\hat{H}$. Similarly, the method followed in deducing the eigenvalues of the generators (self-adjoint) of the 2-dimensional homogeneous Lorentz group and its Casimir operator may be used for the case of $K < 0$ [39]. However, we must remember that the operators arising in our case act on the corresponding $L^2(CC^2)$, while in the two similar cases mentioned above the operators act on different Hilbert space.

In §(8.5.1) we shall evaluate the eigenvalues of the momentum operators in any CC^N. Anticipating the results of those evaluations, we list here the spectra $S(\hat{P}_\mu)$ of the quantum momentum observables in CC^2:

Case I $K < 0$

$$\begin{aligned} S(\hat{P}_1) &= S(\hat{P}_2) = \mathbb{R}\,(\text{gm-cm/sec}), \\ S(\hat{P}_3) &= \hbar\,\mathbb{N}\,(\text{gm-cm}^2/\text{sec}) \quad ; \end{aligned} \tag{3.5.28}$$

Case II $K = 0$

$$\begin{aligned} S(\hat{P}_1) &= S(P_2) = \mathbb{R}\,(\text{gm-cm/sec}), \\ S(\hat{P}_3) &= \hbar\,\mathbb{N}\,(\text{gm-cm}^2/\text{sec}) \quad ; \end{aligned} \tag{3.5.29}$$

Case III $K>0$

$$S(\hat{P}_1) = S(\hat{P}_2) = \sqrt{K}\,\hbar\,\mathbb{N}(\text{gm-cm/sec}), \qquad S(\hat{P}_3) = \hbar\,\mathbb{N}(\text{gm-cm}^2/\text{sec}) \quad . \tag{3.5.30}$$

Here $\mathbb{N}$ denotes the set of integers and $\alpha\mathbb{N}$ the set of numbers which are multiples of integers by a constant α. The above formulae show that the spectra of $\hat{P}_1$ and $\hat{P}_2$ are continuous if $K\leq 0$ and discrete if $K>0$. The spectrum of $\hat{P}_3$ is always discrete.

The spectra of $\hat{P}'_\mu$ given by (3.5.24) are

$$K<0 \quad \begin{cases} S(\hat{P}'_1) = S(\hat{P}'_2) = \mathbb{R}(\text{gm-cm}^2/\text{sec}), \\ S(\hat{P}'_3) = \hbar\,\mathbb{N}\ (\text{gm-cm}^2/\text{sec}) \quad ; \end{cases} \tag{3.5.31}$$

$$K>0 \qquad S(\hat{P}') = \hbar\,\mathbb{N}\ (\text{gm-cm}^2/\text{sec}) \qquad (\mu=1,2,3). \tag{3.5.32}$$

As we have mentioned earlier in this section, we may apply the same methods used to derive the eigenvalues of Casimir operators for SO(3) and SO(2,1) [39] to find the eigenvalues of the Hamiltonian $\hat{H}$ in CC^2_- and CC^2_+. Remembering that $\hat{H}$ in our case is positive-definite since the metric is positive-definite, we may list the spectrum of $\hat{H}$ in all types of CC^2:

$$\begin{aligned} &\text{Case } K\leq 0 \quad S(\hat{H}) = \mathbb{R}^+(\text{erg}), \\ &\text{Case } K>0 \quad S(\hat{H}) = \frac{K}{2m}\,n(n+1)\hbar^2\ (\text{erg}) \quad (n=0,1,2,\dots). \end{aligned} \tag{3.5.33}$$

Here $\mathbb{R}^+$ denotes the set of non-negative real numbers $[0,\infty)$.

§(5.4.2) Physical Distinction of CC^2 with Different Curvatures

CC^2 of different types are physically distinguishable in the sense that measurements of some observables can tell us the type of the space. A CC^2_+ is characterized by the fact that the spectra of the

Hamiltonian and of all the momenta are discrete. Though the Hamiltonian and the momenta $\hat{P}_1$ and $\hat{P}_2$ possess continuous spectra in CC_-^2 and in CC_0^2, and moreover $\hat{P}_3$ has the same spectrum in both types of spaces, a CC_0^2 is singled out from the rest of spaces of constant curvature by the simultaneous measurability of $\hat{P}_1$ and $\hat{P}_2$.

In spite of the common features possessed by CC^2 of the same type, they are still distinguishable in that the value of the curvature does manifest itself.

For the case $K>0$, the curvature K appears in the discrete spectrum of $\hat{H}$. Therefore, energy measurements can tell us the curvature of a CC_+^2 .

When $K<0$, the curvature manifests itself in the expectation values of $\hat{H}$. Consider two CC_-^2 V_1 and V_2 with curvatures K_1 and K_2 and geodesic coordinates (x_1,y_1) and (x_2,y_2) respectively. The coordinates x_i,y_i $(i=1,2)$ range from $-\infty$ to $+\infty$. Let us perform in V_2 the coordinate transformation

$$x_2 = \sqrt{K_1/K_2}\, x_1 \quad , \quad y_2 = \sqrt{K_1/K_2}\, y_1 \tag{3.5.34}$$

The correspondence $(x_1,y_1)\in V_1 \leftrightarrow (x_1,y_1)\in V_2$ is a homeomorphism between V_1 and V_2. This homeomorphism produces a 1-1 onto correspondence $\psi(x_1,y_1)\in \mathbb{C}(V_1)\leftrightarrow \psi(x_1,y_1)\in \mathbb{C}(V_2)$ between $\mathbb{C}(V_1)$ and $\mathbb{C}(V_2)$. Here $\mathbb{C}(V_i)$ denotes the set of complex functions defined on V_i $(i=1,2)$. From (3.5.1a) we have

$$K_2\, ds_2^2 = K_1\, ds_1^2 = K_1(\, dx_1^2 + ch^2\sqrt{-K_1}\, x_1\, dy_1^2\,) \tag{3.5.35}$$

where ds_1^2 and ds_2^2 are the metric forms of V_1 and V_2. Denote the inner product in $L^2(V_i)$ by $(\ ,\ )_i$ $(i=1,2)$. If $\psi(x_1,y_1), \varphi(x_1,y_1)\in L^2(V_1)$, then

$$(\psi,\varphi)_1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \psi^*\varphi\sqrt{g_1}\, dx_1\, dy_1 < \infty ,$$

where g_1 is the determinant of ds_1^2. Now

$$(\psi,\varphi)_2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \psi^*\varphi\sqrt{g_2}\, dx_1\, dy_1 \qquad (g_2 \text{ is the determinant of } ds_2^2)$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \psi^*\varphi\,(K_1/K_2)\sqrt{g_1}\, dx_1\, dy_1 \text{ (by (5.3.35))}$$

$$= (K_1/K_2)(\psi,\varphi)_1 < \infty . \qquad (3.5.36)$$

Therefore the correspondence $\psi \in L^2(V_1) \leftrightarrow \psi \in L^2(V_2)$ is a vector space isomorphism with the property that the inner products in $L^2(V_1)$ and $L^2(V_2)$ as Hilbert spaces satisfy the relation

$$K_1(\ ,\ )_1 = K_2(\ ,\ )_2 . \qquad (3.5.37)$$

In particular

$$\|\psi\|_2 = \sqrt{K_1/K_2}\ \|\psi\|_1 . \qquad (3.5.38)$$

Thus, ignoring a difference by a multiplicative factor between the inner products in $L^2(V_1)$ and $L^2(V_2)$, we may envisage these two Hilbert spaces as being essentially the same space. An operator defined on $L^2(V_1)$ may be looked on as being defined on $L^2(V_2)$. From (3.5.35) we have $\hat{H}_1/K_1 = \hat{H}_2/K_2$, where $\hat{H}_1$ and $\hat{H}_2$ are the Hamiltonian operators in V_1 and V_2 respectively. A state $\psi(x_1,y_1)$ of a quantum system in V_1 may be considered as a state of an identical system in V_2. Let ψ be a state function of a quantum system in V_2 and let us calculate the expectation values of $\hat{H}_2$:

$$\langle H_2\rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \psi^*\hat{H}_2\psi\sqrt{g_2}\, dx_1 dy_1 \Big/ \|\psi\|_2^2$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \psi^*\frac{K_2}{K_1}\hat{H}_1\psi\Big(\frac{K_1}{K_2}\sqrt{g_1}\Big)\, dx_1 dy_1 \Big/ \|\psi\|_2^2$$

$$= \frac{K_2}{K_1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \psi^* \hat{H}_1 \psi \sqrt{g_1}\, dx_1\, dy_1 \Big/ \left(\frac{K_2}{K_1} \|\psi\|_2^2 \right)$$

$$= \frac{K_2}{K_1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \psi^* \hat{H}_1 \psi \sqrt{g_1}\, dx_1\, dy_1 \Big/ \|\psi\|_1^2 .$$

Thus the expectation values of H_1 and H_2 are related by the equation

$$\langle H_2 \rangle = (K_2/K_1) \langle H_1 \rangle \quad . \tag{3.5.39}$$

This shows that we can tell the relative curvature of V_1 and V_2 through energy measurements of two identical systems prepared in the same state.

§(5,4,3) The Equations of Motion

The classical momenta P_μ (μ=1,2,3) are constants of the motion since

$$\dot{P}_\mu = \{P_\mu, H\} = 0.$$

It is known that the integrals of this shorten the calculation when one solves the Hamiltonian equations of motion. As an example we consider the free motion in a CC_-^2 . By (3.5.21a), the equations of motion are

$$\begin{aligned} &m\,\dot{x} = p_x\,, \qquad m\,\dot{y} = p_y / ch^2\sqrt{-K}\,x \quad , \\ &m\,\dot{p}_x = \sqrt{-K}\, th\sqrt{-K}\,x\; p_y^2 / ch^2\sqrt{-K}\,x \;\; , \;\; \dot{p}_y = 0 . \end{aligned} \tag{3.5.40}$$

Now we may use the first two equations of (3.5.16) to evaluate p_x and p_y, then we substitute for their values in the first two equations of (3.5.40) to get the equations of motions in the following form :

$$\begin{aligned} &m\,\dot{x} = (1/ch\sqrt{-K}\,x)\, P_1 + th\sqrt{-K}\,x\; th\sqrt{-K}\,y\;\, P_2 \quad , \\ &m\,\dot{y} = P_2 / ch^2\sqrt{-K}\,y \end{aligned} \tag{3.5.41}$$

The quantal case is similar to the classical one. From (3.5.15a) we find that the Hiesenberg equations of motion are

$$m\dot{\hat{x}} = \hat{p}_x - \frac{i}{2}\hbar\sqrt{-K}\, th\sqrt{-K}\,x \quad , \tag{3.5.42a}$$

$$m\,\dot{\hat{y}} = (1/ch^2\sqrt{-K}\,x)\,\hat{p}_y \quad , \tag{3.5.42b}$$

$$m\dot{\hat{p}}_x = \sqrt{-K}\,(th\sqrt{-K}\,x/ch^2\sqrt{-K}\,x)\,\hat{p}_y^2 - (\tfrac{i}{2}\hbar K/ch^2\sqrt{-K}\,x)\,\hat{p}_x + \tfrac{1}{2}\hbar^2 K\sqrt{-K}\; th\sqrt{-K}\,x/ch^2\sqrt{-K}\,x\,, \quad (3.5.42c)$$

$$\dot{\hat{p}}_y = 0 \quad .$$

Introducing the fact that $\hat{P}_\mu$ (μ=1,2,3) are constants of the motion, we can find the equations of motion of $\hat{x}$ and $\hat{y}$ without involving the p's. From (3.5.9a), (3.5.9b), (3.5.12), (3.5.42a) and (3.5.42b) we find that the equations of motion of $\hat{x}$ and $\hat{y}$ are

$$\begin{aligned} m\dot{\hat{x}} &= (1/ch\sqrt{-K}\,y)\,\hat{P}_1 + th\sqrt{-K}\,x\; th\sqrt{-K}\,y\,\hat{P}_2 - \tfrac{i}{2}\hbar\sqrt{-K}\; th\sqrt{-K}\,x\,, \\ m\dot{\hat{y}} &= (1/ch^2\sqrt{-K}\,x)\,\hat{P}_2\,. \end{aligned} \quad (3.5.43)$$

These equations agree in the classical limit ($\hbar\to 0$) with (3.5.41).

§(5.4.4) The Forms of the Momenta in Geodesic Polar Coordinates

If a geodesic polar coordinate system (r,ϕ) is adopted then an infinitesimal motion is of the form

$$L = (\alpha\cos\phi + a\sin\phi)\,\partial/\partial r + \{\sqrt{-K}\,cth\sqrt{-K}\,r(-\alpha\sin\phi + a\cos\phi) + \gamma\}\,\partial/\partial\phi \quad (3.5.44)$$

in CC_-^2 ,

$$L = (\alpha\cos\phi + a\sin\phi)\,\partial/\partial r + \{\sqrt{K}\cot\sqrt{K}\,r(-\alpha\sin\phi + a\cos\phi) + \gamma\}\,\partial/\partial\phi \quad (3.5.45)$$

in CC_+^2 , and

$$L = (\alpha\cos\phi + a\sin\phi)\,\partial/\partial r + \{\tfrac{1}{r}(-\alpha\sin\phi + a\cos\phi) + \gamma\}\,\partial/\partial\phi \quad (3.5.46)$$

in CC_0^2 .

A choice of three independent quantum momenta could be:

Case $K<0$

$$\begin{aligned} P_1 &= \cos\phi\,\hat{p}_r - \sqrt{-K}\,cth\sqrt{-K}\,r\,\sin\phi\,\hat{p}_\phi\,, \\ P_2 &= \sin\phi\,\hat{p}_r + \sqrt{-K}\,cth\sqrt{-K}\,r\,\cos\phi\,\hat{p}_\phi\,, \\ P_3 &= \hat{p}_\phi\,; \end{aligned} \quad (3.5.47)$$

Case $K>0$

$$\begin{aligned} P_1 &= \cos\phi\,\hat{p}_r - \sqrt{K}\cot\sqrt{K}\,r\,\sin\phi\,\hat{p}_\phi\,, \\ P_2 &= \sin\phi\,\hat{p}_r + \sqrt{K}\cot\sqrt{K}\,r\,\cos\phi\,\hat{p}_\phi\,, \\ P_3 &= \hat{p}_\phi\,; \end{aligned} \quad (3.5.48)$$

Case $K = 0$

$$
\begin{aligned}
P_1 &= \cos\phi\,\hat{p}_r - \frac{1}{r}\sin\phi\,\hat{p}_\phi \quad ,\\
P_2 &= \sin\phi\,\hat{p}_r + \frac{1}{r}\cos\phi\,\hat{p}_\phi \quad ,\\
P_3 &= \hat{p}_\phi \quad .
\end{aligned}
\tag{3.5.49}
$$

These momenta satisfy the commutation relations (3.5.13) and the equation (3.5.14).

§6. Quantization in CC^3.

Using geodesic polar coordinates [35,29], the metric in a CC^3 takes one of the forms [35]

$$ds^2 = dr^2 - \frac{1}{K}\, sh^2\sqrt{-K}\,r\, \sin^2\theta\, d\phi^2 - \frac{1}{K}\, sh^2\sqrt{-K}\,r\, d\theta^2 \quad (K = -\frac{1}{R^2} < 0) \quad (3.6.1a)$$

$$ds^2 = dr^2 + \frac{1}{K}\sin^2\sqrt{K}\,r\,\sin^2\theta\, d\phi^2 + \frac{1}{K}\sin^2\sqrt{K}\,r\, d\theta^2 \quad (K = \frac{1}{R^2} > 0) \quad (3.6.1b)$$

$$ds^2 = dr^2 + r^2\sin^2\theta\, d\phi^2 + r^2 d\theta^2 \qquad (K = 0) \qquad (3.6.1c)$$

§(6.1) The Infinitesimal Motions of CC^3,

§(6.1.1) The Case of CC^3_- .

From (3.6.1a) we find the following six Killing equations:

$$\left.\begin{array}{ll}
\xi^1_{,1} = 0 & , \ \mathrm{I} \\
\xi^2_{,2} + \sqrt{-K}\, cth\sqrt{-K}\,r\, \xi^1 + \cot\theta\, \xi^3 = 0 & , \ \mathrm{II} \\
\xi^3_{,3} + \sqrt{-K}\, cth\sqrt{-K}\,r\, \xi^1 = 0 & , \ \mathrm{III} \\
\xi^1_{,2} - \frac{1}{K}\, sh^2\sqrt{-K}\,r\, \sin^2\theta\, \xi^2_{,1} = 0 & , \ \mathrm{IV} \\
\xi^1_{,3} - \frac{1}{K}\, sh^2\sqrt{-K}\,r\, \xi^3_{,1} = 0 & , \ \mathrm{V} \\
\sin^2\theta\, \xi^2_{,3} + \xi^3_{,2} = 0 & , \ \mathrm{VI}
\end{array}\right\} \quad (3.6.2)$$

Equation I shows that ξ^1 is independent of r, and hence

$$\xi^1 = \xi^1(\phi, \theta) \qquad (3.6.3)$$

Substituting for ξ^1 in III and integrating with respect to θ, we find

$$\xi^3 = -\sqrt{-K}\, cth\sqrt{-K}\,r \int_0^\theta \xi^1(\phi,\theta)\, d\theta + f(r,\phi), \qquad (3.6.4)$$

where $f(r,\phi)$ is a function of r and ϕ is to be determined. Substituting for ξ^1 and ξ^3 in V from (3.6.4) and (3.6.3), we get an equation which may be put in the form

$$\frac{\partial \xi^1(\phi,\theta)}{\partial \theta} + \int_0^\theta \xi^1(\phi,\theta)\, d\theta = \frac{1}{K}\, sh^2\sqrt{-K}\, r\, \frac{\partial f(r,\phi)}{\partial r}. \tag{3.6.5}$$

The left-hand side of this equation is a function of ϕ and θ, while its right-hand side is a function of r and ϕ, If we fix ϕ, then the left-hand side of (3.6.5) is a function of θ, while the right-hand side is a function of r. Since r and θ vary independently, both sides must equal a common constant g which varies only with ϕ. Thus, we have

$$\frac{\partial \xi^1(\phi,\theta)}{\partial \theta} + \int_0^\theta \xi^1(\phi,\theta)\, d\theta = g(\phi), \tag{3.6.6}$$

$$\frac{\partial f(r,\phi)}{\partial r} = \frac{K g(\phi)}{sh^2\sqrt{-K}\, r}. \tag{3.6.7}$$

The differentiation of (3.6.6) with respect to θ gives

$$\frac{\partial^2 \xi(\phi,\theta)}{\partial \theta^2} + \xi^1(\phi,\theta) = 0. \tag{3.6.8}$$

Now, we seek solutions for eq.(3.3.8) of the form

$$\xi^1(\phi,\theta) = Y(\phi)\, Z(\theta), \tag{3.6.9}$$

where Y and Z are functions of ϕ and θ respectively. The substitution for ξ^1 from (3.6.9) in (3.6.8) gives

$$\frac{d^2 Z(\theta)}{d\theta^2} + Z(\theta) = 0.$$

The general solution of this equation is

$$Z(\theta) = A\cos\theta + B\sin\theta, \tag{3.6.10}$$

where A and B are arbitrary constants. From (3.6.9) and (3.6.10) we have

$$\xi^1(\phi,\theta) = Y(\phi)(A\cos\theta + B\sin\theta), \tag{3.6.11}$$

where $Y(\phi)$ is to be determined. Substituting for $\xi^1(\phi,\theta)$ from (3.6.11) in (3.6.6), we get

$$Y(\phi)(-A\sin\theta + B\cos\theta) + Y(\phi)\left[A\sin\theta - B\cos\theta\right]_0^\theta = g(\phi)$$

$$\Rightarrow g(\phi) = B\, Y(\phi). \tag{3.6.12}$$

By (3.6.12), eq. (3.6.7) takes the form

$$\frac{\partial f(r,\phi)}{\partial r} = \frac{KBY(\phi)}{sh^2\sqrt{-K}\,r}.$$

Integrating the above equation with respect to r, we get

$$f(r,\phi) = \sqrt{K}\;BY(\phi)\,cth\sqrt{-K}\,r + h(\phi), \tag{3.6.13}$$

where $h(\phi)$ is a function of ϕ to be determined. Substituting for $\xi^1(\phi,\theta)$ and $f(r,\phi)$ in (3.6.4) from (3.6.11) and (3.6.13), we find

$$\xi^3 = -\sqrt{-K}\,cth\sqrt{-K}\,r\,Y(\phi)\left[A\sin\theta - B\cos\theta\right]_0^\theta + \sqrt{-K}\,BY(\phi)\,cth\sqrt{-K}\,r + h(\phi)$$

$$\Rightarrow \xi^3 = -\sqrt{-K}\,cth\sqrt{-K}\,r\,Y(\phi)\,(A\sin\theta - B\cos\theta). \tag{3.6.14}$$

The substitution for ξ^1 from (3.6.11) in IV gives

$$\frac{dY(\phi)}{d\phi}(A\cos\theta + B\sin\theta) - \frac{1}{K}sh^2\sqrt{-K}\,r\,\sin^2\theta\,\xi^2_{,1} = 0$$

On solving the above equation for $\xi^2_{,1}$, we find

$$\xi^2_{,1} = \frac{K}{sh^2\sqrt{-K}\,r}\;\frac{A\cos\theta + B\sin\theta}{\sin^2\theta}\;\frac{dY(\phi)}{d\phi}.$$

Integrating this last equation with respect to r, we get

$$\xi^2 = -\sqrt{-K}\,cth\sqrt{-K}\,r\;\frac{A\cos\theta + B\sin\theta}{\sin^2\theta}\;\frac{dY(\phi)}{d\phi} + \Psi(\phi,\theta). \tag{3.6.15}$$

The differentiation of (3.6.15) (3.6.14) with respect to θ and ϕ respectively gives

$$\xi^2_{,3} = \sqrt{-K}\,cth\sqrt{-K}\,r\,\frac{dY(\phi)}{d\phi}\left(-A\frac{1+\cos^2\theta}{\sin^2\theta} - B\frac{\cos\theta}{\sin^2\theta}\right) + \frac{\partial\Psi(\phi,\theta)}{\partial\theta} = 0,$$

$$\xi^3_{,2} = -\sqrt{-K}\,cth\sqrt{-K}\,r\,\frac{dY(\phi)}{d\phi}(A\sin\theta - B\cos\theta) + \frac{dh(\phi)}{d\phi} = 0.$$

After substituting for $\xi^2_{,3}$ and $\xi^3_{,2}$ from these last two equations in VI and cancelling the appropriate terms we find an equation which may be put in the following form:

$$\sin^2\theta\,\frac{\partial\Psi(\phi,\theta)}{\partial\theta} + \frac{dh(\phi)}{d\phi} = 2A\sqrt{-K}\,cth\sqrt{-K}\,r\,\frac{dY(\phi)}{d\phi}\,\frac{1}{\sin\theta}.$$

Solving this equation means finding functions $\psi(\phi,\theta)$, $h(\phi)$ and $Y(\phi)$ so that this equation becomes an identity in the variables involved when ψ, h and Y are substituted by their functional forms. But the left-hand side does not depend on r while the right-hand side does. Thus, both sides of the above equation must vanish, yielding

$$-2A\sqrt{-K}\ \mathrm{cth}\sqrt{-K}\,r\ \frac{dY(\phi)}{d\phi}\ \frac{1}{\sin\theta} = 0, \tag{3.6.16}$$

$$\frac{\partial\psi(\phi,\theta)}{\partial\theta} = -\frac{1}{\sin^2\theta}\ \frac{dh(\phi)}{d\phi}. \tag{3.6.17}$$

Integrating (3.6.17) with respect to θ, we find

$$\psi(\phi,\theta) = \cot\theta\ \frac{dh(\phi)}{d\phi} + G(\phi), \tag{3.6.18}$$

where $G(\phi)$ is a function of ϕ to be determined. Turning back to (3.6.16) we encounter two possibilities:

$$\text{(i) } A=0\ ,\ \text{(ii) } \frac{dY(\phi)}{d\phi}=0.$$

We consider each of these possibilities separately.

(i) If A=0, then equations (3.6.11), (3.6.15) and (3.6.14) after making use of (3.6.18) take the form

$$\xi^1(\phi,\theta) = B\,Y(\phi)\sin\theta \qquad , \tag{3.6.19}$$

$$\xi^2(r,\phi,\theta) = \frac{B\sqrt{-K}\ \mathrm{cth}\sqrt{-K}\,r}{\sin\theta}\ \frac{dY(\phi)}{d\phi} + \cot\theta\ \frac{dh(\phi)}{d\phi} + G(\phi)\ , \tag{3.6.20}$$

$$\xi^3(r,\phi,\theta) = B\sqrt{-K}\ \mathrm{cth}\sqrt{-K}\,r\ Y(\phi)\cos\phi + h(\phi) \qquad . \tag{3.6.21}$$

The constant B is redundant since it is mingled with the undetermined function $Y(\phi)$, and it can be dropped. Substituting for ξ^1, ξ^2 and ξ^3 from these above three equations in II, we find an equation which may be put in the form

$$\frac{\sqrt{-K}\ \mathrm{cth}\sqrt{-K}\,r}{\sin\theta}\left(\frac{d^2Y(\phi)}{d\phi^2} + Y(\phi)\right) + \cot\theta\left(\frac{d^2h(\phi)}{d\phi^2} + h(\phi)\right) + \frac{dG(\phi)}{d\phi} = 0. \tag{3.6.22}$$

This equation is satisfied iff

$$\frac{d^2Y(\phi)}{d\phi^2} + Y(\phi) = 0,$$

$$\frac{d^2h(\phi)}{d\phi^2} + h(\phi) = 0,$$

$$\frac{dG(\phi)}{d\phi} = 0.$$

The general solutions of these equations are

$$\begin{aligned} Y(\phi) &= \alpha \cos\phi + a \sin\phi\,, \\ h(\phi) &= \beta \cos\phi + b \sin\phi\,, \\ G(\phi) &= \gamma\,, \end{aligned} \tag{3.6.23}$$

where α, β, γ, a and b are arbitrary constants. Thus the general solution of the system I-VI of partial differential equations under the possibility (i) is found through substituting for Y, h and G from (3.6.23) in (3.6.19) and (3.6.21). the result is

$$\xi^1 = \sin\theta\,(\alpha \cos\phi + a \sin\phi),$$

$$\xi^2 = \frac{\sqrt{-K}\,\mathrm{cth}\sqrt{-K}\,r}{\sin\theta}(-\alpha \sin\phi + a \cos\phi) + \cot\theta(-\beta \sin\phi + b \cos\phi) + \gamma\,, \tag{3.6.24}$$

$$\xi^3 = \sqrt{-K}\,\mathrm{cth}\sqrt{-K}\,r \cos\theta\,(\alpha \cos\phi + a \sin\phi) + (\beta \cos\phi + b\sin\phi).$$

From (3.6.24) we can deduce only five independent Killing vectors since the number of arbitrary constants involved is only five, and hence we expect to find a sixth Killing vector which is independent of these previous five in the solution of the system I-VI under the possibility (ii).

(ii) If $\frac{dY(\phi)}{d\phi} = 0$, then Y=k (an arbitrary constant). In this case we can write (3.6.11), (3.6.15) and (3.6.14), after making use of (3.6.18), in the form

$$\begin{aligned} \xi^1(\theta) &= A \cos\theta + B \sin\theta \\ \xi^2(\phi,\theta) &= \cot\theta \frac{dh(\phi)}{d\phi} + G(\phi) \\ \xi^3(r,\phi,\theta) &= -\sqrt{-K}\,\mathrm{cth}\sqrt{-K}\,r\,(A \sin\theta - B \cos\theta) + h(\phi), \end{aligned} \tag{3.6.25}$$

having absorbed the constant k into A and B. Substituting for ξ^1, ξ^2

and ξ^3 from (3.6.25) in II and making suitable cancellation, we find

$$\cot\theta\left(\frac{d^2h(\phi)}{d\phi^2}+h(\phi)\right)+\frac{dG(\phi)}{d\phi}+\frac{B\sqrt{-K}\,cth\sqrt{-K}r}{\sin\theta}=0\ .$$

This above equation is satisfied iff

$$B=0\ ,\quad \frac{dG(\phi)}{d\phi}=0\ ,\quad \frac{d^2h(\phi)}{d\phi^2}+h(\phi)=0\ .$$

Thus

$$B=0\ ,\ G=\gamma'\ ,\ h(\phi)=\beta'\cos\phi+b'\sin\phi\ . \tag{3.6.26}$$

where γ', β' and b' are arbitrary constants. The general solution of (I-VI) under the possibility (ii) is found through substituting for B, G and $h(\phi)$ from (3.6.26) in (3.6.25). The result is

$$\begin{aligned}\xi^1 &= A\cos\theta\ ,\\ \xi^2 &= \cot\theta(-\beta'\sin\theta+b\cos\theta)+\gamma'\ ,\\ \xi^3 &= -A\sqrt{-K}\,cth\sqrt{-K}r\sin\theta+\beta'\cos\phi+b'\sin\phi\ .\end{aligned} \tag{3.6.27}$$

From (3.6.27) we may obtain four independent Killing vectors. However, only one of them is independent of those arising from (3.6.24). The simplest choice of such a new Killing vector is

$$\xi^1=A\cos\theta\ ,\ \xi^2=0\ ,\ \xi^3=-A\sqrt{-K}\,cth\sqrt{-K}r\sin\theta\ . \tag{3.6.28}$$

Combining (3.6.24) and (3.6.28) we can make the following statement : the general infinitesimal motion of a CC^3_- is of the form $\xi^1\partial/\partial r+\xi^2\partial/\partial\phi+\xi^3\partial/\partial\theta$, where

$$\begin{aligned}\xi^1 &= \sin\theta(\alpha\cos\phi+a\sin\phi)+A\cos\theta\ ,\\ \xi^2 &= \sqrt{-K}\,cth\sqrt{-K}r(-\alpha\sin\phi+a\cos\phi)/\sin\theta\\ &\quad+\cot\theta(-\beta\sin\phi+b\cos\phi)+\gamma\ ,\\ \xi^3 &= \sqrt{-K}\,cth\sqrt{-K}r(\alpha\cos\phi+a\sin\phi)\cos\theta\\ &\quad-A\sqrt{-K}\,cth\sqrt{-K}r\sin\theta+(\beta\cos\phi+b\sin\phi)\ .\end{aligned} \tag{3.6.29}$$

§(6.1.2) The Case of CC^3_+

Following a similar method to that of (6.1.1) we find that the general form of an infinitesimal motion of a CC_+^3 is $\xi^1 \partial/\partial r + \xi^2 \partial/\partial\phi + \xi^3 \partial/\partial\theta$, where

$$\begin{aligned}
\xi^1 &= \sin\theta\,(\alpha\cos\phi + a\sin\phi) + A\cos\theta \quad , \\
\xi^2 &= \sqrt{K}\cot\sqrt{K}\,r\,(-\alpha\sin\phi + a\cos\phi)/\sin\theta \\
&\quad + \cot\theta\,(-\beta\sin\phi + b\cos\phi) + \gamma \quad , \\
\xi^3 &= \sqrt{K}\cot\sqrt{K}\,r\,(\alpha\cos\phi + a\sin\phi)\cos\theta \\
&\quad - A\cot\sqrt{K}\,r\sin\theta + (\beta\cos\phi + b\sin\phi) \, .
\end{aligned} \tag{3.6.30}$$

§(6.1.3) The Case of E^3

This may be considered as a limit of either of the previous cases. Letting $K \to 0$ in either (3.6.29) or (3.6.30) we find that the general form of an infinitesimal motion of E^3 is $\xi^1 \partial/\partial r + \xi^2 \partial/\partial\phi + \xi^3 \partial/\partial\theta$, where

$$\begin{aligned}
\xi^1 &= \sin\theta\,(\alpha\cos\phi + a\sin\phi) + A\cos\theta \quad , \\
\xi^2 &= \frac{1}{r\sin\theta}(-\alpha\sin\phi + a\cos\phi) + \cot\theta(-\beta\sin\phi + b\cos\phi) + \gamma \, , \\
\xi^3 &= \frac{1}{r}(\alpha\cos\phi + a\sin\phi)\cos\theta - \frac{A}{r}\sin\theta + (\beta\cos\phi + b\sin\phi) \, .
\end{aligned} \tag{3.6.31}$$

§(6.2) The Momenta and the Hamiltonian

By suitably choosing the constants in eqs. (3.6.29), (3.6.30), (3.6.31) we obtain the following independent infinitesimal motions :

Case I $K<0$

$$\begin{aligned}
L_1 &= \cos\phi\sin\phi\,\partial/\partial r - \sqrt{-K}\,\frac{\mathrm{cth}\sqrt{-K}\,r\sin\phi}{\sin\theta}\,\partial/\partial\phi + \sqrt{-K}\,\mathrm{cth}\sqrt{-K}\,r\cos\phi\cos\theta\,\partial/\partial\theta \, , \\
L_2 &= \sin\phi\sin\theta\,\partial/\partial r + \sqrt{-K}\,\frac{\mathrm{cth}\sqrt{-K}\,r\cos\phi}{\sin\theta}\,\partial/\partial\phi + \sqrt{-K}\,\mathrm{cth}\sqrt{-K}\,r\sin\phi\cos\theta\,\partial/\partial\theta \, , \\
L_3 &= \cos\theta\,\partial/\partial r - \sqrt{-K}\,\mathrm{cth}\sqrt{-K}\,r\sin\theta\,\partial/\partial\theta \quad , \\
L_4 &= \cos\phi\cot\theta\,\partial/\partial\phi + \sin\phi\,\partial/\partial\theta \quad , \\
L_5 &= -\sin\phi\cot\theta\,\partial/\partial\phi + \cos\phi\,\partial/\partial\theta \quad , \\
L_6 &= \partial/\partial\phi \quad ;
\end{aligned} \tag{3.6.32}$$

Case II $K > 0$

$$
\begin{aligned}
L_1 &= \cos\phi \sin\theta\, \partial/\partial r - \sqrt{K}\, \frac{\cot\sqrt{K}r \sin\phi}{\sin\theta}\, \partial/\partial\phi + \sqrt{K} \cot\sqrt{K}r \cos\phi \cos\theta\, \partial/\partial\theta, \\
L_2 &= \sin\phi \sin\theta\, \partial/\partial r + \sqrt{K}\, \frac{\cot\sqrt{K}r \cos\phi}{\sin\theta}\, \partial/\partial\phi + \sqrt{K} \cot\sqrt{K}r \sin\phi \cos\theta\, \partial/\partial\theta, \\
L_3 &= \cos\theta\, \partial/\partial r - \sqrt{K} \cot\sqrt{K}r\, \sin\theta\, \partial/\partial\theta \;, \\
L_4 &= \cos\phi \cot\theta\, \partial/\partial\phi + \sin\phi\, \partial/\partial\theta \;, \\
L_5 &= -\sin\phi \cot\theta\, \partial/\partial\phi + \cos\phi\, \partial/\partial\theta \;, \\
L_6 &= \partial/\partial\phi \;;
\end{aligned}
\tag{3.6.33}
$$

Case III $K = 0$

$$
\begin{aligned}
L_1 &= \cos\phi \sin\theta\, \partial/\partial r - \frac{\sin\phi}{r\sin\theta}\, \partial/\partial\phi + \frac{1}{r} \cos\phi \cos\theta\, \partial/\partial\theta \;, \\
L_2 &= \sin\phi \sin\theta\, \partial/\partial r + \frac{\cos\phi}{r\sin\theta}\, \partial/\partial\phi + \frac{1}{r} \sin\phi \cos\theta\, \partial/\partial\theta \;, \\
L_3 &= \cos\phi\, \partial/\partial r - \frac{1}{r} \sin\theta\, \partial/\partial\theta \;, \\
L_4 &= \cos\phi \cot\theta\, \partial/\partial\phi + \sin\phi\, \partial/\partial\theta \;, \\
L_5 &= -\sin\phi \cot\theta\, \partial/\partial\phi + \cos\phi\, \partial/\partial\theta \;, \\
L_6 &= \partial/\partial\phi \;.
\end{aligned}
\tag{3.6.34}
$$

The corresponding quantum observables are simply

$$\hat{P}_\mu = -i\hbar L_\mu \qquad (\mu = 1, \ldots, 6) \tag{3.6.35}$$

If we denote L_μ by

$$L_\mu = \xi^1_\mu\, \partial/\partial r + \xi^2_\mu\, \partial/\partial\phi + \xi^3_\mu\, \partial/\partial\theta \;,$$

then the corresponding classical observables are

$$p_\mu = \xi^1_\mu p_r + \xi^2_\mu p_\phi + \xi^2_\mu p_\theta \;,$$

where p_r, p_ϕ, p_θ are the classical generalized momenta conjugate to r, ϕ and θ respectively.

It can be verified in every case that the quantum Hamiltonian is related to the momentum operators by the relation

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 = \frac{1}{2m} (\hat{P}_1^2 + \hat{P}_2^2 + \hat{P}_3^2 + K(\hat{P}_4^2 + \hat{P}_5^2 + \hat{P}_6^2)), \tag{3.6.36}$$

and explicitly

$$H = -\frac{\hbar^2}{2m}\left\{\frac{\partial^2}{\partial r^2} - \frac{K}{sh^2\sqrt{-K}\,r}\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\theta^2}\right) + 2\sqrt{-K}\,cth\sqrt{-K}\,r\frac{\partial}{\partial r} - \frac{K\cot\theta}{sh^2\sqrt{-K}\,r}\frac{\partial}{\partial\theta}\right\} \quad \text{if } K<0, \tag{3.6.37a}$$

$$= -\frac{\hbar^2}{2m}\left\{\frac{\partial^2}{\partial r^2} + \frac{K}{\sin^2\sqrt{K}\,r}\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\theta^2}\right) + 2\sqrt{K}\cot\sqrt{K}\,r\frac{\partial}{\partial r} + \frac{K\cot\theta}{\sin^2\sqrt{K}\,r}\frac{\partial}{\partial\theta}\right\} \quad \text{if } K>0, \tag{3.6.37b}$$

$$= -\frac{\hbar^2}{2m}\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\theta^2}\right) + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\cot\theta\frac{\partial}{\partial\theta}\right\} \quad \text{if } K=0. \tag{3.6.37c}$$

Let

$$\hat{N}_1 = -\hat{P}_1, \quad \hat{N}_2 = -\hat{P}_2, \quad \hat{N}_3 = -\hat{P}_3;$$
$$\hat{M}_1 = -\hat{P}_4, \quad \hat{M}_2 = \hat{P}_5, \quad \hat{M}_3 = \hat{P}_6.$$

It can be verified in every case that the quantum momentum operators satisfy the commutation relations

$$\begin{aligned} [\hat{N}_i, \hat{N}_j] &= i\hbar K\,\varepsilon_{ijk}\hat{N}_k, \\ [\hat{M}_i, \hat{M}_j] &= i\hbar\,\varepsilon_{ijk}\hat{M}_k, \\ [\hat{M}_i, \hat{N}_j] &= [\hat{N}_i, \hat{M}_j] = i\hbar\,\varepsilon_{ijk}\hat{N}_k, \end{aligned} \tag{3.6.38}$$

where ε_{ijk} is the permutation symbol. In terms of $\hat{M}_i$ and $\hat{N}_j$, the relation (3.6.36) may be written as

$$\hat{H} = \frac{1}{2m}\left(\hat{\underline{N}}^2 + K\,\hat{\underline{M}}^2\right) \tag{3.6.39}$$

where

$$\hat{\underline{N}}^2 = \hat{N}_1^2 + \hat{N}_2^2 + \hat{N}_3^2, \quad \hat{\underline{M}}^2 = \hat{M}_1^2 + \hat{M}_2^2 + \hat{M}_3^2.$$

Classically, the following relation holds :

$$H = \frac{1}{2m}\left(P_1^2 + P_2^2 + P_3^2 + K(P_4^2 + P_5^2 + P_6^2)\right), \tag{3.6.40}$$

and explicitly

$$\hat{H} = \frac{1}{2m}\left\{p_r^2 - \frac{K}{sh^2\sqrt{-K}\,r}\left(\frac{p_\phi^2}{\sin^2\theta} + p_\theta^2\right)\right\} \qquad \text{if } K<0$$

$$= \frac{1}{2m}\left\{p_r^2 + \frac{K}{\sin^2\sqrt{K}\,r}\left(\frac{p_\phi^2}{\sin^2\theta} + p_\theta^2\right)\right\} \qquad \text{if } K>0 \qquad (3.6.41)$$

$$= \frac{1}{2m}\left\{p_r^2 + \frac{1}{r^2}\left(\frac{p_\phi^2}{\sin^2\theta} + p_\theta^2\right)\right\} \qquad \text{if } K=0 \ .$$

also if M_i and N_i are the corresponding classical momenta to the quantal momenta M_i and N_i, then

$$\begin{aligned} \{N_i, N_j\} &= K\,\varepsilon_{ijk}M_k \ , \\ \{M_i, M_j\} &= \varepsilon_{ijk}M_k \ , \\ \{M_i, N_j\} &= \{N_i, M_j\} = \varepsilon_{ijk}N_k \ . \end{aligned} \qquad (3.6.42)$$

If we define new quantum momenta $\hat{N}_i'$ and $\hat{M}_i'$ by

$$\hat{N}_i' = \hat{N}_i/\sqrt{eK} \ , \quad \hat{M}_i' = \hat{M}_i \quad (eK = |K|) \ , \qquad (3.6.43)$$

then the new momenta satisfy the commutation relations

$$\begin{aligned} [\hat{N}_i', \hat{N}_j'] &= e\,\varepsilon_{ijk}\,\hat{M}_k' \\ [\hat{M}_i', \hat{M}_j'] &= \varepsilon_{ijk}\,\hat{M}_k' \\ [\hat{M}_i', \hat{N}_j'] &= [\hat{N}_i', \hat{M}_j'] = \varepsilon_{ijk}\,\hat{N}_k' \ . \end{aligned} \qquad (3.6.44)$$

The Hamiltonian in terms of $\hat{M}_i'$ and $\hat{N}_i'$ may be written in the form

$$\hat{H} = \frac{eK}{2m}\left(\underline{\hat{N}}'^2 + e\,\underline{\hat{M}}'^2\right) . \qquad (3.6.45)$$

§(6.3) Physical Considerations

We shall find in §(8.5.1) that the spectra of the quantum momenta in various types of CC^3 are

$$\begin{aligned} S(\hat{N}_i) &= \sqrt{K}\,\hbar\,\mathbb{N}(\text{gm-cm/sec}) \\ S(\hat{M}_i) &= \hbar\,\mathbb{N}(\text{gm-cm}^2/\text{sec}) \end{aligned} \qquad \text{if } K>0 \ , \qquad (3.6.46)$$

$$\begin{aligned} S(\hat{N}_i) &= \mathbb{R}\ (\text{gm-cm/sec}) \\ S(\hat{M}_i) &= \hbar\,\mathbb{N}(\text{gm-cm}^2/\text{sec}) \end{aligned} \qquad \text{if } K\leq 0 \ . \qquad (3.6.47)$$

One can see that when $K=1$ the commutation relations (3.6.38) are identical with those of the generators of the 4-dimensional rotation group [37]. A similar statement holds for for every space with $K>0$ since the K explicitly contained in (3,6,38) may be absorbed into $\hat{N}_i$. For the negative curvature space $K = -1$, (3,6,38) are essentially identical with those for the generators of the 3-dimensional homogeneous Lorentz group [37]. This is true for every space with $K<0$.

These observations mean that the properties of our present momentum observables and the Hamiltonian may be worked out readily from the knowledge of SO(4) and SO(3,1).

§7. Detailed Study of the Hyperbolic Plane and the Hyperbolic Space

The hyperbolic plane may stand as a model for the hyperbolic geometry (Lobachefsky's geometry) in the same sense as the Euclidean plane and the 2-sphere may stand as models for the Euclidean and Riemannian (elliptic) geometries respectively [36]. Contrary to the case of a CC_+^2 which could be embedded as a sphere in E^3, there is a well-known theorem by Hilbert which states that a CC_-^2 cannot be embedded as a regular surface in E^3 [36]. In the course of our study of the hyperbolic plane we will discuss its geometric features and study the dynamics in it in detail. The results obtained can be carried over almost completely to all CC_-^2 and moreover can be generalized when one considers CC_-^3.

§(7.1) Definition and Geometrical Properties

Let M be the upper half of $\mathbb{R}^2$. i.e., $M = \{(x,y)\in\mathbb{R}^2 \mid y>0\}$. M is coverable by a single coordinate chart $\varphi: M \to \mathbb{R}\times(0,\infty)$. Define a metric on M by

$$ds^2 = (dx^2 + dy^2)/y^2. \qquad (3.7.1)$$

The Riemannian manifold M with the metric (3.7.1) is of constant curvature $K = -1$ and is called the hyperbolic plane [36].

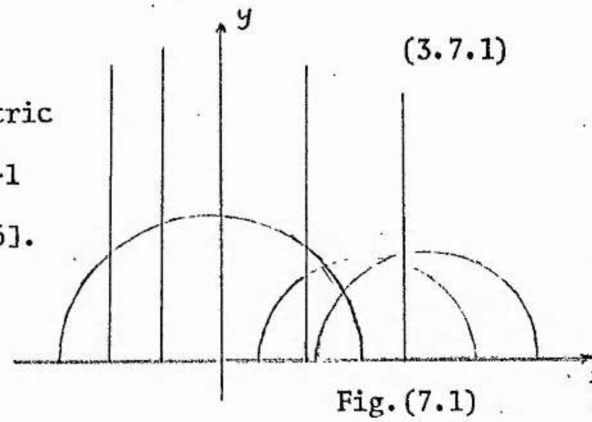


Fig. (7.1)

The 1-parameter family of straight lines parallel to oy ($x=x_0$) and the 2-parameter family of half-circles with centres on the x-axis and arbitrary radii ($(x-x_0)^2+y^2=a^2$) furnish all geodesics of the hyperbolic plane. We note that the words "straight lines" and "circles" are used figuratively (Fig. (7.1)). As is the case in the Euclidean plane, it can be shown that the geodesic lines in the hyperbolic plane are infinitely long in both directions [36]. i.e., the hyperbolic plane is a complete Riemannian manifold [32.p154]. If we take the straight lines of the hyperbolic plane to be its geodesics, then it is not true that parallels to a geodesic line through a point m not on it are uniquely determined. For instance, $x = 0$ is a geodesic, but if $m(x,y) \in M$ with $x \neq 0$, then it is clear that one can draw an infinite number of half-circles passing through m and not intersecting the line $x = 0$. All such half-circles are parallel to $x = 0$ since they do not have any point in common with it. For more illustrative discussion of the above concepts we refer to Stoker [36,p185].

§(7,2) The Momenta and the Hamiltonian

The general form of an infinitesimal motion of the hyperbolic plane [App. 5] is

$$L = \left\{ \frac{\alpha}{2}(x^2-y^2) + \beta x + \gamma \right\} \partial/\partial x + (\alpha x + \beta) y \, \partial/\partial y , \qquad (3.7.2)$$

where α, β and γ are arbitrary constants. A choice of three independent infinitesimal motions may be

$$\begin{aligned} L_1 &= x \, \partial/\partial x + y \, \partial/\partial y , \\ L_2 &= \tfrac{1}{2}(x^2-y^2-1) \partial/\partial x + xy \, \partial/\partial y , \qquad (3.7.3) \\ L_3 &= \tfrac{1}{2}(x^2-y^2+1) \partial/\partial x + xy \, \partial/\partial y . \end{aligned}$$

The corresponding quantum momentum observables are

$$\hat{P}_\mu = -i\hbar L_\mu \quad (\mu = 1,2,3) \qquad (3.7.4)$$

These momenta satisfy the commutation relations

$$[\hat{P}_1, \hat{P}_2] = i\hbar K \hat{P}_3 \; , \; [\hat{P}_2, \hat{P}_3] = i\hbar \hat{P}_1 \; , \; [\hat{P}_3, \hat{P}_1] = i\hbar \hat{P}_2 \qquad (3.7.5)$$

and the relation

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 = \frac{1}{2m} (\hat{P}_1^2 + \hat{P}_2^2 + K \hat{P}_3^2) \qquad (3.7.6a)$$

$$= -\frac{\hbar^2}{2m}y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right). \qquad (3.7.6b)$$

The classical momenta are

$$\begin{aligned} P_1 &= x\,p_x + y\,p_y , \\ P_2 &= \tfrac{1}{2}(x^2-y^2-1)\,p_x + xy\,p_y , \\ P_3 &= \tfrac{1}{2}(x^2-y^2+1)\,p_x + xy\,p_y . \end{aligned} \qquad (3.7.7)$$

The classical momenta satisfy the Poisson bracket relations

$$\{P_1,P_2\} = K\,P_3 \ , \ \{P_2,P_3\} = P_1 \ , \ \{P_3,P_1\} = P_2 \ , \qquad (3.7.8)$$

and the relation

$$H = \frac{1}{2m}\left(P_1^2 + P_2^2 + K\,P_3^2 \right) = \frac{1}{2m}\,y^2\left(p_x^2 + p_y^2 \right) , \qquad (3.7.9)$$

Remark : We may start with another set of independent infinitesimal motions and obtain three quantum momenta. These commute with H but do not necessarily satisfy (3.7.6). For instance, the quantum momenta corresponding to the infinitesimal motions

$$L'_1 = x\,\partial/\partial x + y\,\partial/\partial y \ , \ L'_2 = \partial/\partial x \ , \ L'_3 = \tfrac{1}{2}(x^2-y^2)\,\partial/\partial x + xy\,\partial/\partial y , \qquad (3.7.10)$$

satisfy the commutation relations

$$[\hat{P}'_1,\hat{P}'_2] = i\hbar\hat{P}'_2 \ , \ [\hat{P}'_2,\hat{P}'_3] = -i\hbar\hat{P}'_1 \ , \ [\hat{P}'_3,\hat{P}'_1] = i\hbar\hat{P}'_3 . \qquad (3.7.11)$$

Hence, $\hat{H}$ written in terms of $\hat{P}'_\mu$ cannot have the form (3.7.9). However the expression of $\hat{H}$ in terms of $\hat{P}'_\mu$ is easily obtained. Since

$$\hat{P}_1 = \hat{P}'_1 \ , \ \hat{P}_2 = \hat{P}'_3 - \tfrac{1}{2}\hat{P}'_2 \ , \ \hat{P}_3 = \hat{P}'_3 + \tfrac{1}{2}\hat{P}'_2 \qquad (3.7.12)$$

the substitution for $\hat{P}_\mu$ from these relations in (3.7.6a) gives

$$\hat{H} = (1/2m)\left(\hat{P}'^2_1 - \hat{P}'_2\,\hat{P}'_3 - \hat{P}'_3\,\hat{P}'_2 \right), \qquad (3.7.13)$$

which is the required expression.

§(7.3) Motions of the Hyperbolic Plane

In this section we consider the set of infinitesimal motions (3.7.3), find their integral curves and determine the OPG's corresponding to each of them. We need all this later in the solution of the eigenvalues of $\hat{P}_\mu$. There follows a summary of relevant results.

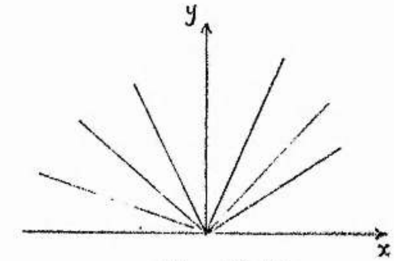


Fig (7.2)

(i) $L_1 = x\ \partial/\partial x + y\ \partial/\partial y$

The integral curves are the family of straight lines $\bar{y} = (y/x)\ \bar{x}$ (Fig.(7.2)). The OPG is

$$\bar{x} = x\, e^t\ ,\quad \bar{y} = y\, e^t\ . \qquad (3.7.14)$$

Here x and y are the initial values of the variables $\bar{x}$ and $\bar{y}$. It is clear from (3.7.14) that

$$t \to -\infty \Rightarrow (x,y) \to (0,0) \text{ and } t \to +\infty \Rightarrow (x,y) \overset{\to\ (\infty,\infty)}{\quad} (x\infty,+\infty).$$

(ii) $L_2 = \frac{1}{2}(x^2 - y^2 - 1)\ \partial/\partial x + xy\ \partial/\partial y$.

The integral curves are the family of incomplete circles

$$x^2 + (y-c)^2 = 1+c^2 \qquad (3.7.15)$$

with centres $(0,c)$ and radii $(1+c^2)^{\frac{1}{2}}$ (Fig.(7.3)). All these incomplete circles pass through the points $(-1,0)$ and $(1,0)$. Since these two points are at infinite distance from any point in the hyperbolic plane, we may say that the family (3.7.15) meet at infinity. The OPG is

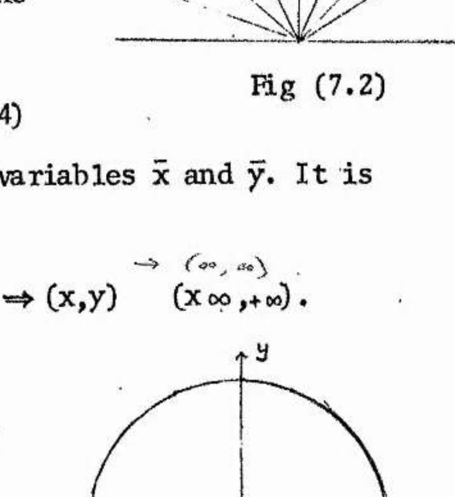


Fig.(7.3)

$$\bar{x}(t) = \frac{2\gamma e^{-t}(\gamma e^{-t} - 2c)}{(\gamma e^{-t} - 2c)^2 + 4} - 1 \qquad , \ \bar{y}(t) = \frac{4\gamma e^{-t}}{(\gamma e^{-t} - 2c)^2 + 4}\ , \quad (3.7.16)$$

where

$$\gamma = [(x+1)^2 + y^2]/y \quad , \qquad 2c = (x^2 + y^2 - 1)/y\ . \qquad (3.7.17)$$

From (3.7.16) it is clear that

$$t \to -\infty \Rightarrow (x,y) \to (1,0) \quad \text{and } t \to +\infty \Rightarrow (x,y) \to (-1,0)\ .$$

(iii) $L_3 = \frac{1}{2}(x^2 - y^2 + 1)\ \partial/\partial x + xy\ \partial/\partial y$

The integral curves of L_3 are the family of circles

$$x^2+(y-c)^2 = c^2-1 \quad (c \geqslant 1) \qquad (3.7.18)$$

with centres $(0,c)$ and radii $(c^2-1)^{\frac{1}{2}}$ (Fig(7.4)). The OPG is

$$\bar{x}(t) = \frac{\sqrt{c^2-1}\ \cos(t-\gamma)}{c-\sqrt{c^2-1}\ \sin(t-\gamma)} ,$$

$$(3.7.19)$$

$$\bar{y}(t) = \frac{1}{c-\sqrt{c^2-1}\ \sin(t-\gamma)} ,$$

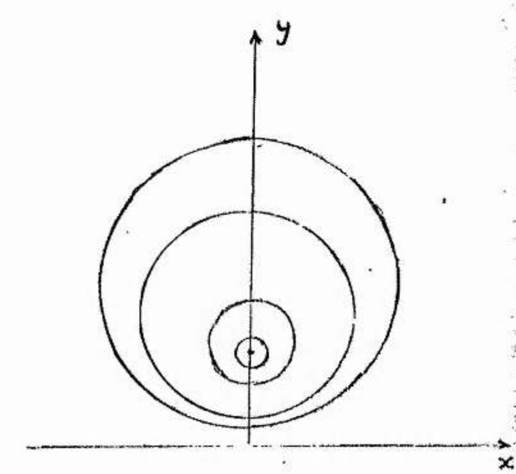


Fig.(7.3)

where

$$2c = \frac{x^2+y^2+1}{y} , \qquad e^{-i\gamma} = \frac{2x+i(x^2+y^2-1)}{\sqrt{(x^2+y^2-1)^2+4x^2}} . \qquad (3.7.20)$$

From (3.7.19) we deduce that the integral curves of L_3 are periodic with period $T = 2\pi$.

§(7.4) Eigenvalues and Eigenfunctions of the Momentum Operators

In order to evaluate the eigenvalues of the momenta $\hat{P}_\mu = -i\hbar L_\mu$ ($\mu=1,2,3$) we modify two propositions given in [33.p146] so that they become applicable to our present case.

Proposition (7,1) If σ is an integral curve of a vector field on a differentiable manifold M and if $\psi \in \mathbb{C}(M)$ is a differentiable function, then

$$\frac{d}{dt}(\sigma^*\psi) = \sigma^*(L\psi) . \qquad (3.7.21)$$

Proof: By the definition of the differential of a map [ch.I.§(1.6)], we find

$$\left(\frac{d}{dt}(\sigma^*\psi)\right)(t) = \left(\sigma_*\left(\frac{d}{dt}\right)_t\right)\psi = L_{\sigma(t)}\psi = (L\psi)(\sigma(t)) = (\sigma^*(L\psi))(t),$$

for every $t \in D_{\sigma^*\psi}$. ∎

Proposition (7.2) A non-zero function ψ is an eigenfunction of the operator $-i\hbar L$ and λ is the corresponding eigenvalue iff

$$\sigma^*\psi = (\sigma^*\psi)(0)\, e^{i\lambda t/\hbar} . \tag{3.7.22}$$

Proof: If $-i\hbar L\psi = \lambda\psi$ for a given complex number λ, it follows from the previous proposition that

$$-i\hbar \frac{d}{dt}(\sigma^*\psi) = \sigma^*(-i\hbar L\psi) = \lambda(\sigma^*\psi),$$

and hence

$$\sigma^*\psi = (\sigma^*\psi)(0)\, e^{i\lambda t/\hbar} .$$

Conversely, if (3.7.22) is true for an integral curve σ defined on the domain I, then

$$\begin{aligned} -i\hbar L(\sigma^*\psi) &= -i\hbar \frac{d}{dt}((\sigma^*\psi)(0)\, e^{i\lambda t/\hbar}) \\ &= \lambda(\sigma^*\psi)(0)\, e^{i\lambda t/\hbar} \\ &= \lambda\, \sigma^*\psi , \end{aligned}$$

and hence

$$-i\hbar L\psi = \lambda\psi \text{ on } \sigma(I).$$

If (3.7.22) is true for every integral curve, then $-i\hbar L\psi = \lambda\psi$.

In our present case the vector fields L_μ (μ=1,2,3) are complete, and hence the momenta $\hat{P}_\mu$ are self-adjoint. It follows that their eigenvalues are real. If the integral curves of a vector field L_μ are periodic with period T, then by (3.7.22) every eigenfunction ψ of L_μ satisfies the condition

$$(\sigma^*\psi)(0) = (\sigma^*\psi)(0)\, e^{i\lambda T/\hbar} .$$

Therefore

$$\lambda = n\hbar\ (2\pi/T) \qquad \text{(n is an integer).} \tag{3.7.23}$$

If the integral curves of L_μ are not periodic, then λ can be any real number since no boundary condition is imposed on ψ.

Applying these last conclusions to our present operators $\hat{P}_1$, $\hat{P}_2$ and $\hat{P}_3$, we find that $\hat{P}_1$ and $\hat{P}_2$ possess continuous spectra ranging from $-\infty$ to $+\infty$, while $\hat{P}_3$ possesses the discrete spectrum $\{\ldots, -2\hbar, -\hbar, 0, \hbar, 2\hbar, \ldots\}$ since its integral curves are periodic with period $T = 2\pi$.

An equivalent way to this discussed above is to calculate the eigenfunctions explicitly utilizing the available boundary conditions. This, of course, yields the same results obtained in the previous paragraph. Eigenfunctions of the momentum operators in an N-dimensional space of constant curvature are evaluated in [App.9].

Denote the eigenfunctions of a momentum operator $\hat{P}$ by $\psi_\rho(\hat{P})$ so that $\hat{P}\,\psi_\rho(\hat{P}) = \rho\,\psi_\rho(P)$. Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. The eigenfunctions of the momentum operators given by (3.7.3) and (3.7.4) are

$$\psi_\lambda(P_1) = y^{i\lambda/\hbar}\, F_1\left(\frac{y}{x}\right) ,$$

$$\psi_\lambda(P_2) = \left\{\frac{y}{(x+1)^2 + y^2}\right\}^{i\lambda/\hbar} F_2\left(\frac{x^2+y^2-1}{y}\right) ,$$

$$\psi_n(P_3) = \left\{\frac{2x + i(x^2+y^2-1)}{\sqrt{4x^2 + (x^2+y^2-1)^2}}\right\}^n F_3\left(\frac{x^2+y^2+1}{y}\right) ,$$

where F_μ (μ=1,2,3) are arbitrary functions in their arguments.

§(7.5) Various Considerations

§(7.5.1) The Equations of Motion

The Hamiltonian equations of motion are

$$m\dot{x} = y^2 p_x \ , \ m\dot{y} = y^2 p_y \ , \ \dot{p}_x = 0 \ , \ m\dot{p}_y = -y(p_x^2 + p_y^2) \, . \tag{3.7.24}$$

We may recover the Lagrangian equations of motion through eliminating p_x and p_y from (3.7.24) to get

$$m\ddot{x} = 2\dot{x}\dot{y}/y \quad , \quad m\ddot{y} = (\dot{y}^2 - \dot{x}^2)/y \, .$$

Alternatively, we make use of (3.7.7) to deduce from (3.7.24) that the equations of motion are

$$m\dot{x} = y^2 p_x \quad , \quad m\dot{y} = -xy\, p_x + y\, P_1 \, . \tag{3.7.25}$$

Observe that p_x and P_1 are constants.

The quantum mechanical case is quite similar. The Hiesenberg equations of motion are

$$\begin{aligned} i\hbar\dot{\hat{x}} &= [\hat{x}, \hat{H}] = \hat{y}^2 \hat{p}_x/m \ , \quad i\hbar\dot{\hat{y}} = [\hat{x}, \hat{H}] = \hat{y}^2 \hat{p}_y/m \ , \\ i\hbar\dot{\hat{p}}_x &= [\hat{p}_x, \hat{H}] = 0 \qquad , \quad i\hbar\dot{\hat{p}}_y = [\hat{p}_y, \hat{H}] = -\hat{y}(\hat{p}_x^2 + \hat{p}_y^2)/m \, . \end{aligned} \tag{3.7.26}$$

We may eliminate $\hat{p}_x$ and $\hat{p}_y$ from (3.7.26) to find

$$m\ddot{\hat{x}} = \dot{\hat{x}}\frac{1}{\hat{y}}\dot{\hat{y}} + \dot{\hat{y}}\frac{1}{\hat{y}}\dot{\hat{x}} \ , \ m\ddot{\hat{y}} = \dot{\hat{y}}\frac{1}{\hat{y}}\dot{\hat{y}} - \dot{\hat{x}}\frac{1}{\hat{y}}\dot{\hat{x}} \ ;$$

or we may utilize (3.7.3) to find the equations of motion of $\hat{x}$ and $\hat{y}$ without involving the generalized momenta. The result is

$$m\dot{\hat{x}} = \hat{y}^2 \hat{p}_x \ , \ m\dot{\hat{y}} = -\hat{x}\hat{y}\hat{p}_x + \hat{y}\hat{P}_1 \, .$$

§(7.5.2) Coordinate Transformations between Various Coordinate Systems in the Hyperbolic Plane

The coordinate transformation between the present coordinate system and a polar coordinate system with pole at $(x=0, y=1)$ and the angle ϕ measured from the geodesic $x^2+y^2=1$ (Fig.(7.5)) is given by

$$x = \operatorname{sh} r \cos\phi/(\operatorname{ch} r - \operatorname{sh} r \sin\phi) \ , \ y = 1/(\operatorname{ch} r - \operatorname{sh} r \sin\phi) \, . \tag{3.7.28}$$

Denote the geodesic parallel coordinate system used in §5 by

(X,Y). The coordinate transformation between (x,y) and (X,Y) with origin at (x=0,y=1) and X =0 coinciding with x =0 (Fig.(7.6)) is given by

$$x = e^{Y}\, th\, X \,, \quad y = e^{Y}/ch\, X \,. \tag{3.7.29}$$

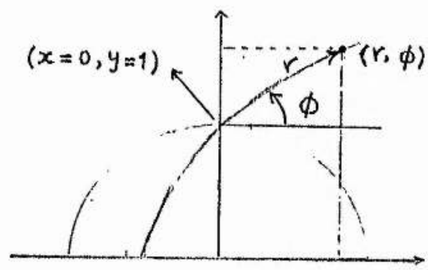


Fig. (7.5)

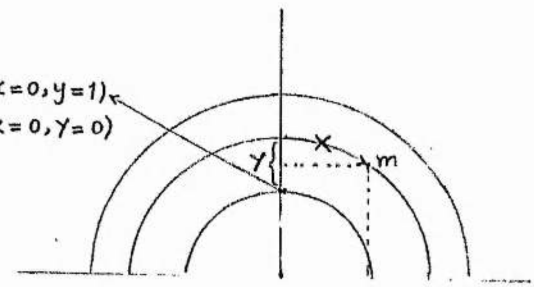


Fig. (7.6)

The transformation (3.7.29) is 1-1 globally. However the transformation (3.7.28) is 1-1 in the domain of definition of the chart (r,ϕ). The deduction of (3.7.28) and (3.7.29) will become clear from the considerations of §(3.1.1) in the next chapter.

§(7.5.3) Generalization to CC^2_-

The manifold $M = \mathbb{R} \times (0,\infty)$ with the metric $ds^2 = R^2(dx^2+dy^2)/y^2$ is of constant curvature $K = -1/R^2$. It is obvious that the infinitesimal motions of any CC^2_- may be chosen to be the same as those (3.7.3) of the hyperbolic plane. Thus all studies concerning their integral curves, the OPG's which they generate, their eigenvalues and their eigenfunctions are applicable here. The Hamiltonian in the present case is related to the momenta through

$$\hat{H} = (-K/2m)\ (\ \hat{P}_1^2 + \hat{P}_2^2 - \hat{P}_3^2\ )\,. \tag{3.7.30}$$

§(7.6) The Hyperbolic Space

Consider a manifold M consisting of all points of the upper half of $\mathbb{R}^3$, i.e. $M = \{(x,y,z) \in \mathbb{R}^3 | z > 0\}$, and endowed with the metric

$$ds^2 = (dx^2 + dy^2 + dz^2)/(-Kz^2) , \tag{3.7.31}$$

where $K < 0$. M is a space of constant curvature equal to K [34]. We shall call this manifold the hyperbolic space.

A general infinitesimal motion of M [App.5] is of the form

$$L = \{ \frac{\alpha_1}{2}(x^2 - y^2 - z^2) + \alpha_2 xy + \beta x + ky + \gamma_1 \} \partial/\partial x$$
$$+ \{ \frac{\alpha_2}{2}(y^2 - x^2 - z^2) + \alpha_1 xy + \beta y - kx + \gamma_2 \} \partial/\partial y + z(\alpha_1 x + \alpha_2 y + \beta)\partial/\partial z . \tag{3.7.32}$$

A choice of six independent infinitesimal motions may be

$$\begin{aligned}
L_1 &= yx\,\partial/\partial x + \tfrac{1}{2}(y^2 - x^2 - z^2 - 1)\partial/\partial y + yz\,\partial/\partial z ,\\
L_2 &= -\tfrac{1}{2}(x^2 - y^2 - z^2 - 1)\partial/\partial x - xy\,\partial/\partial y - xz\,\partial/\partial z ,\\
L_3 &= x\,\partial/\partial x + y\,\partial/\partial y + z\,\partial/\partial z ,\\
L_4 &= \tfrac{1}{2}(x^2 - y^2 - z^2 + 1)\partial/\partial x + xy\,\partial/\partial y + xz\,\partial/\partial z ,\\
L_5 &= yx\,\partial/\partial x + \tfrac{1}{2}(y^2 - x^2 - z^2 + 1)\partial/\partial y + yz\,\partial/\partial z ,\\
L_6 &= x\,\partial/\partial y - y\,\partial/\partial x .
\end{aligned} \tag{3.7.33}$$

The OPG's of motions $U_t^{\mu}(x,y,z) = (\bar{x}(x,y,z,t), \bar{y}(x,y,z,t), \bar{z}(x,y,z,t))$ generated by L_μ are [App.6]

(i)

$$U^1 \equiv \begin{cases} \bar{x} = \dfrac{4\alpha\gamma e^{-t}}{(\gamma e^{-t} - \beta)^2 + 4(1+\alpha^2)} , \\[2ex] \bar{y} = \dfrac{2\gamma e^{-t}(\gamma e^{-t} - \beta)}{(\gamma e^{-t} - \beta)^2 + 4(1+\alpha^2)} - 1 , \\[2ex] \bar{z} = \dfrac{4\gamma e^{-t}}{(\gamma e^{-t} - \beta)^2 + 4(1+\alpha^2)} , \end{cases} \tag{3.7.34}$$

where

$$\alpha = x/z\ ,\ \beta = (x^2+y^2+z^2-1)/z\ ,\ \gamma = [x^2+(y+1)^2+z^2]/z\ . \tag{3.7.35}$$

(ii)

$$U^2 \equiv \begin{cases} \bar{x} = \dfrac{2\gamma e^t(\gamma e^t-\beta)}{(\gamma e^t-\beta)^2+4(1+\alpha^2)} - 1\ , \\ \bar{y} = \dfrac{4\alpha\gamma e^t}{(\gamma e^t-\beta)^2+4(1+\alpha^2)}\ , \\ \bar{z} = \dfrac{4\gamma e^t}{(\gamma e^t-\beta)^2+4(1+\alpha^2)}\ , \end{cases} \tag{3.7.36}$$

where

$$\alpha = y/z\ ,\ \beta = (x^2+y^2+z^2-1)/z\ ,\ \gamma = [(x+1)^2+y^2+z^2]/z\ . \tag{3.7.37}$$

(iii) $$U^3(x,y,z) = (xe^t, ye^t, ze^t) \tag{3.7.38}$$

(iv)

$$U^4 \equiv \begin{cases} \bar{x} = \dfrac{\sqrt{\beta^2-4(1+\alpha^2)}\cos(t-\gamma)}{\beta-\sqrt{\beta^2-4(1+\alpha^2)}\sin(t-\gamma)}\ , \\ \bar{y} = \dfrac{2\alpha}{\beta-\sqrt{\beta^2-4(1+\alpha^2)}\sin(t-\gamma)}\ , \\ \bar{z} = \dfrac{2}{\beta-\sqrt{\beta^2-4(1+\alpha^2)}\sin(t-\gamma)}\ , \end{cases} \tag{3.7.39}$$

where

$$\alpha = y/z\ ,\ \beta = (x^2+y^2+z^2+1)/z\ ,$$
$$e^{-i\gamma} = \{2x+i(x^2+y^2+z^2-1)\}/\sqrt{4x^2+(x^2+y^2+z^2-1)}\ . \tag{3.7.40}$$

(v)

$$U^5 \equiv \begin{cases} \bar{x} = \dfrac{2\alpha}{\beta-\sqrt{\beta^2-4(1+\alpha^2)}\sin(t-\gamma)}\ , \\ \bar{y} = \dfrac{\sqrt{\beta^2-4(1+\alpha^2)}\cos(t-\gamma)}{\beta-\sqrt{\beta^2-4(1+\alpha^2)}\sin(t-\gamma)}\ , \end{cases} \tag{3.7.41}$$

$$\bar{z} = \frac{2}{\beta - \sqrt{\beta^2 - 4(1+\alpha^2)}\ \sin(t-\gamma)} ,$$

where

$$\alpha = x/z \ , \ \beta = (x^2+y^2+z^2+1)/z \ ,$$

$$e^{-i\gamma} = \{2y + i(x^2+y^2+z^2-1)\}/\sqrt{4y^2+(x^2+y^2+z^2-1)} \ . \qquad (3.7.42)$$

(vi) $U^6(x,y,z) = ((x^2+y^2)^{\frac{1}{2}}\cos(t-\gamma), (x^2+y^2)^{\frac{1}{2}}\sin(t-\gamma), z)$ (3.7.43)

where

$$e^{-i\gamma} = (x+iy)/\sqrt{x^2+y^2} \ . \qquad (3.7.44)$$

If we denote the orbits of U^μ by σ^μ, then such orbits are given by [App.6]

(i)
$$\sigma^1 : \begin{cases} \bar{x} = \alpha \bar{z} & , \\ \bar{x}^2 + \bar{y}^2 + \bar{z}^2 - 1 = \beta \bar{z} & . \end{cases} \qquad (3.7.45)$$

We notice that all σ^1 pass through (0,1,0) and (0,-1,0). From (3.7.34) we have

$$t \to \pm\infty \Rightarrow (x,y,z) \to (0,\mp 1,0). \qquad (3.7.46)$$

(ii)
$$\sigma^2 : \begin{cases} \bar{y} = \alpha \bar{z} & , \\ \bar{x}^2 + \bar{y}^2 + \bar{z}^2 - 1 = \beta \bar{z} & . \end{cases} \qquad (3.7.47)$$

All σ^2 pass through (1,0,0) and (-1,0,0). From (3.7.36) we have

$$t \to \pm\infty \Rightarrow (x,y,z) \to (\pm 1,0,0). \qquad (3.7.48)$$

(iii) $\sigma^3 : \bar{x}/x = \bar{y}/y = \bar{z}/z$. (3.7.49)

All σ^3 pass through (0,0,0), and from (3.7.38) we have

$$\begin{aligned} t &\to -\infty \Rightarrow (x,y,z) \to (0,0,0) , \\ t &\to +\infty \Rightarrow (x,y,z) \to (x_\infty, y_\infty, \infty) . \end{aligned} \qquad (3.7.50)$$

In all the above three cases the points through which the orbits pass are not in the manifold and actually never reached as (3.7.46), (3.7.48) and (3.7.50) show. We may say in each case that σ^{μ} (μ=1,2,3) meet at infinity.

(iv)
$$\sigma^4: \quad \begin{array}{l} \bar{y} = \alpha\,\bar{z} \,, \\ \bar{x}^2 + \bar{y}^2 + \bar{z}^2 + 1 = \beta\,\bar{z} \,. \end{array} \tag{3.7.51}$$

σ^4 are the intersections of planes through ox with spheres of radii $(\beta^2/4 - 1)^{\frac{1}{2}}$ and centres $(0,0,\beta/2)$. σ^4 never intersect with the plane $z = 0$. From (3.7.51) we deduce that

$$\beta \geqslant 2(1+\alpha^2)^{\frac{1}{2}} \Rightarrow \inf \beta = 2.$$

Therefore σ^4 are closed curves, Moreover, by (3.7.39), σ^4 are periodic with a common period $T = 2\pi$. When $\alpha = 0$ and $\beta = 2$, the corresponding orbit degenerates to the point (0,0,1).

(v)
$$\sigma^5: \quad \begin{array}{l} \bar{z} = \alpha\,\bar{x} \,, \\ \bar{x}^2 + \bar{y}^2 + \bar{z}^2 + 1 = \beta\,\bar{z} \,. \end{array} \tag{3.7.52}$$

σ^5 are circles formed of the intersections of spheres of radii $(\beta^2/4 - 1)^{\frac{1}{2}}$ and centres $(0,0,\beta/2)$ with planes through oy. From (3.7.52) we have $\beta \geqslant 2(1+\alpha^2)^{\frac{1}{2}}$. By (3.7.41), σ^5 are periodic with a common period $T = 2\pi$. When $\alpha = 0$ and $\beta = 2$, the orbit degenerates to the point (0,0,1). We note that in (iv) and (v), the centres and radii of σ^4 and σ^5 are fictitious.

(vi)
$$\sigma^6: \quad \begin{array}{l} \bar{z} = z \,, \\ \bar{x}^2 + \bar{y}^2 = a^2 \,. \end{array} \tag{3.7.53}$$

σ^6 are circles with centres $(0,0,z)$ and radii a. By (3.7.43), σ^6 are periodic with a common period $T = 2\pi$. The z-axis is a set of fixed points of the OPG U^6 corresponding to $a = 0$.

The quantum momenta corresponding to L_μ ($\mu=1,\ldots,6$) are

$$\hat{N}_i = -i\hbar L_i \quad , \quad \hat{M}_i = -i\hbar L_{i+3} \quad (i=1,2,3) \tag{3.7.54}$$

These momenta satisfy the commutation relations

$$\begin{aligned} [\hat{N}_i,\hat{N}_j] &= -i\hbar\, \varepsilon_{ijk}\, \hat{M}_k \ , \\ [\hat{M}_i,\hat{M}_j] &= i\hbar\, \varepsilon_{ijk}\, \hat{M}_k \ , \\ [\hat{N}_i,\hat{M}_j] &= [\hat{M}_i,\hat{N}_j] = i\hbar\, \varepsilon_{ijk}\, \hat{N}_k \ . \end{aligned} \tag{3.7.55}$$

Thus K has been absorbed in the $\hat{N}$'s [26]. However, K still appears in the Hamiltonian as the following relation shows

$$\hat{H} = (-K/2m)(\underline{\hat{N}}^2 - \underline{\hat{M}}^2) \ . \tag{3.7.56}$$

Similar relations to (3.7.55) and (3.7.56) hold classically when $\hat{N}_i$ and $\hat{M}_i$ are replaced by their corresponding classical momenta and the commutators are replaced by Poisson brackets.

The eigenfunctions of the momentum operators are

$$\psi_\lambda(P_1) = \{ z/[x^2+(y+1)^2+z^2]\}^{i\lambda/\hbar} F(x/z, (x^2+y^2+z^2-1)/z) \quad (\lambda \in \mathbb{R}),$$

$$\psi_\lambda(P_2) = \{[(x+1)^2+y^2+z^2]/z\}^{i\lambda/\hbar} F(y/z, (x^2+y^2+z^2-1)/z) \quad (\lambda \in \mathbb{R}),$$

$$\psi_\lambda(P_3) = z^{i\lambda/\hbar} F(x/z, y/z) \quad (\lambda \in \mathbb{R}),$$

$$\psi_n(P_4) = \left\{ \frac{2x + i(x^2+y^2+z^2-1)}{\sqrt{4x^2+(x^2+y^2+z^2-1)^2}} \right\}^n F\left(\frac{y}{z}, \frac{x^2+y^2+z^2+1}{z}\right) \quad (n \in \mathbb{N}),$$

$$\psi_n(P_5) = \left\{ \frac{2y + i(x^2+y^2+z^2-1)}{\sqrt{4y^2+(x^2+y^2+z^2-1)^2}} \right\}^n F\left(\frac{x}{z}, \frac{x^2+y^2+z^2+1}{z}\right) \quad (n \in \mathbb{N}),$$

$$\psi_n(P_6) = \left\{ \frac{x+iy}{\sqrt{x^2+y^2}} \right\}^n F(x^2+y^2, z) \quad (n \in \mathbb{N}),$$

where F is an arbitrary function in its arguments.

The spectra of the momentum operators are

$$S(\hat{N}_1) = S(\hat{N}_2) = S(\hat{N}_3) = \mathbb{R}\ (\text{gm-cm}^2/\text{sec}) \ ,$$
$$S(\hat{M}_1) = S(\hat{M}_2) = S(\hat{M}_3) = \hbar\mathbb{N}\ (\text{gm-cm}^2/\text{sec}) \ .$$

§8. Quantization in CC^N

In this section we present a unified treatment of the subject of quantization in spaces of constant curvature of any dimension.

§(8.1) Geometrical Considerations

Consider a CC^N with a system of coordinates $(x^1, \ldots, x^N)$ in terms of which the metric assumes the Riemannian form (3.4.2). If $K \geqslant 0$, then

$$-\infty < x^i < +\infty \quad (i=1, \ldots, N) \tag{3.8.1}$$

If $K < 0$, then x^i $(i=1, \ldots, N)$ are restricted by the relation

$$x^i x^i < -4/K , \tag{3.8.2}$$

and hence

$$\sup x^i = 2/\sqrt{-K} . \tag{3.8.3}$$

For a CC^N_+ the system $(x^1, \ldots, x^N)$ cannot be global (since we are considering compact CC^N_+). However, we will see soon that such a system can cover the whole of a CC^N_+ apart from a point. In down-to-earth language, this point which cannot be covered by such a system is just the pole O' opposite to the pole $O(0, \ldots, 0)$. For CC^N_- and CC^N_0 , the above system is global.

In order to familiarize ourselves with such a coordinate system, we consider the 2-dimensional case starting with CC^2_+ . Let V_K be a CC^2_+ with curvature K. In terms of geodesic polar coordinates (r, ϕ), the metric form of V_K is

$$ds^2 = dr^2 + \frac{1}{K}\sin^2\sqrt{K}\, r\, d\phi^2 \tag{3.8.4}$$

The coordinate transformation between the above two systems is

$$x^1 = \frac{2}{\sqrt{K}} \tan\frac{\sqrt{K}}{2} r \cos\phi , \quad x^2 = \frac{2}{\sqrt{K}} \tan\frac{\sqrt{K}}{2} r \sin\phi . \tag{3.8.5}$$

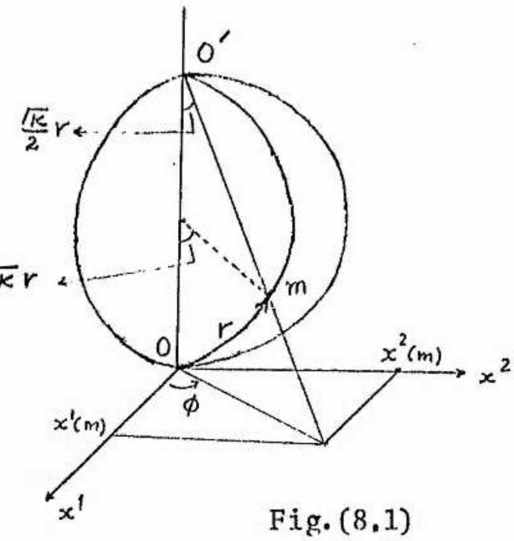


Fig. (8.1)

(See Fig. (8.1)). The transformation (3.8.5) maps $V_K - \{O'\}$ homeomorphically on $\mathbb{R}^2$; it is called the stereographic projection of V_K onto the plane [40]. It is obvious from (3.8.5) that when $r \to \pi/\sqrt{K}$ the coordinates x^1 and x^2 tend to infinity. Hence all points of V_K with coordinates (x^1, x^2) which are infinitely large in absolute value are infinitesimally close to each other. Since $\phi = \phi_0$ ($\phi_0 \in [0, 2\pi)$) are geodesics and therefore r is the geodesic distance from O to any point (r, ϕ_0), we deduce that the parametric equations of geodesics of V_K which pass through O are given by

$$x^1 = \frac{2}{\sqrt{K}} \cos\alpha \tan\frac{\sqrt{K}}{2} s, \qquad x^2 = \frac{2}{\sqrt{K}} \sin\alpha \tan\frac{\sqrt{K}}{2} s, \qquad (3.8.6)$$

where the parameter s is the arc length along each geodesic and $\alpha \in [0, 2\pi)$. We may say that the geodesics which pass through O are just the family of straight lines $x^1 = \beta x^2$ (β is an arbitrary constant).

If V_K is a CC^N_- with curvature K, then any point at any distance from O can be covered by the system (x^1, x^2). Actually this is clear from the metric form, or from the coordinate transformation

$$x^1 = \frac{2}{\sqrt{-K}} \operatorname{th} \frac{\sqrt{-K}}{2} r \cos\phi, \qquad x^2 = \frac{2}{\sqrt{-K}} \operatorname{th} \frac{\sqrt{-K}}{2} r \sin\phi, \qquad (3.8.7)$$

between the geodesic polar system (r, ϕ) and the system (x^1, x^2). Equations (3.8.7) are simply the parametric equations of geodesics of V_K which pass through O with r as the geodesic distance along each geodesic $\phi = \phi_0$. If $r \to \pm\infty$, then $(x^1, x^2) \to (\frac{2}{\sqrt{-K}} \cos\phi_0, \frac{2}{\sqrt{-K}} \sin\phi_0)$. Thus if $m \in V_K$ lies at an arbitrary distance r from O in a direction which makes an angle ϕ_0 with the geodesic $x^1 = 0$, then there exists (x^1, x^2) with $-2/\sqrt{-K} < x^1, x^2 < 2/\sqrt{-K}$ so that $d(O,m) = r$. Such x^1 and x^2 are given by (3.8.7).

The subject dealt with above may be extended easily to any dimension. One can show that the system $(x^1, \ldots, x^N)$ is global if $K \leqslant 0$, and it covers the whole of a CC_+^N apart from the pole O' opposite to the pole $O(0, \ldots, 0)$. This can be achieved through introducing a geodesic polar coordinate system in CC^N, then carrying through a similar argument to this discussed above. This system of geodesic coordinates is discussed in [App.7].

§(8.2) Infinitesimal Motions of CC^N

In terms of the coordinates system (3.4.2) in a CC^N, the Killing equations can be written in the form

$$\xi^i_{,i} = Kx^r\xi^r/2(1+\frac{K}{4}x^rx^r) \quad (r,i=1,\ldots,N;\text{ no summation over } i),$$

$$\xi^i_{,j} + \xi^j_{,i} = 0 \quad (i \neq j; i,j=1,\ldots,N) \cdot$$

These Killing equations give rise to $N(N+1)/2$ independent infinitesimal motions. A choice of such may be ([App.1], [40.p326])

$$L_i = \left\{\frac{K}{4}\Big[(x^i)^2 - \sum_{\substack{j=1\\ j\neq i}}^{N}(x^j)^2\Big] + 1\right\}\partial/\partial x + \frac{K}{2}x^i\sum_{\substack{j=1\\ j\neq i}} x^j\partial/\partial x^j \text{ (i is not summed)}, \tag{3.8.8}$$

$$L_{rs} = x^r\partial/\partial x^s - x^s\partial/\partial x^r,$$

where $(i,s,r = 1, \ldots, N; s > r)$.

These fields satisfy the commutation relations

$$[L_i, L_j] = -K\, L_{ij}\ ,$$
$$[L_i, L_{ij}] = L_j \quad (i \neq j),\quad [L_i, L_{jr}] = 0 \quad (i \neq j, i \neq r),$$
$$[L_{ij}, L_{jk}] = L_{ik} \quad (i \neq j, k \neq j),\quad [L_{ij}, L_{rk}] = 0 \ (i \neq r, i \neq k; j \neq r, j \neq k).$$

(In order to cover all the commutation relations between the infinitesimal motions, we have dispensed in the above formulae with the convention $i<j$ in L_{ij}). For the needs of the next section we order these infinitesimal motions in the following way:

$$\begin{aligned} L_1, \ldots, L_N, L_{N+1} = L_{12}, L_{N+2} = L_{13}, \ldots, L_{2N-1} &= L_{1\,N} \\ L_{2N} = L_{23}, \ldots, L_{3N-3} &= L_{2\,N} \\ &\vdots \\ L_{N(N+1)/2} &= L_{N-1\,N} \end{aligned}$$

In other words we adhere to the following conventions :

(i) The order of L_k is k.

(ii) The index m in L_{mn} is always less than n.

(iii) The order of L_{mn} is

$$\begin{aligned} (mn) &= N + (N-1) + (N-2) + \ldots + (N-(m-1)) + n - m \\ &= Nm - m(m+1)/2 + n\ . \end{aligned}$$

It is clear that for any number $N < b \leq N(N+1)/2$, there correspond unique m and n such that $m<n$ and $m, n \in \{1, \ldots N\}$. i.e., if $(mn) = (rs)$, then $m = r$ and $n = s$.

Under these conventions, the commutation relations are

$$\begin{aligned} [L_m, L_n] &= (-K\,\delta^r_m \delta^s_n + K\,\delta^r_n \delta^s_m)\, L_{(rs)} \\ [L_m, L_{(rs)}] &= (\delta^m_r \delta^k_s - \delta^m_s \delta^k_r)\, L_k = -[L_{(rs)}, L_m], \qquad (3.8.9) \\ [L_{(mn)}, L_{(rs)}] &= (\delta^{nij}_{rms} - \delta^{mij}_{srn} - \delta^{nij}_{smr} + \delta^{nij}_{srm} - \delta^{mij}_{rns} + \delta^{mij}_{rsn})\, L_{(ij)}\ , \end{aligned}$$

where $\delta^{abc}_{tuv} = \delta^a_t \delta^b_u \delta^c_v$.

§(8.3) Motions of CC^N and their Orbits

Let $U^{ij}(x^1, \dots, x^N) = (\bar{x}^1(x^1,\dots,x^N,t), \dots, \bar{x}^N(x^1,\dots,x^N,t))$ be the OPG generated by L_{ij} . Each motion U^{ij} ($i \neq j; i,j=1,\dots,N$) is given by

$$\begin{aligned} &\bar{x}^r = x^r \quad (r \neq i, r \neq j) , \\ &\bar{x}^i = ((x^i)^2 + (x^j)^2)^{\frac{1}{2}} \cos(t - t_0), \\ &\bar{x}^j = ((x^i)^2 + (x^j)^2)^{\frac{1}{2}} \sin(t - t_0), \end{aligned} \tag{3.8.10}$$

where

$$\cos t_0 = x^i/((x^i)^2 + (x^j)^2)^{\frac{1}{2}}, \quad \sin t_0 = -x^j/((x^i)^2 + (x^j)^2)^{\frac{1}{2}}. \tag{3.8.11}$$

Let us adhere to the following convention : When the index s of any quantity q^s appearing in (3.8.12) - (3.8.19) exceeds N, then q^s is identified with q^{s-N}. Now, the groups of motions U^i generated by L_i ($i=1, \dots ,N$) are given by

Case $K < 0$

$$\begin{aligned} &\bar{x}^i = -\frac{\sqrt{-K}}{4} \frac{\gamma(\alpha - \gamma e^{\sqrt{-K}t}) e^{\sqrt{-K}t}}{-\frac{K}{16}(\alpha - \gamma e^{\sqrt{-K}t})^2 + 1 + \sum_{r=i+2}^{i+N-1} \beta_r^2} - \frac{2}{\sqrt{-K}} \\ &\bar{x}^{i+1} = \frac{\gamma e^{\sqrt{-K}t}}{-\frac{K}{16}(\alpha - \gamma e^{\sqrt{-K}t})^2 + 1 + \sum_{r=i+2}^{i+N-1} \beta_r^2} \\ &\bar{x}^r = \beta_r \bar{x}^{i+1} \qquad (r=i+2,\dots,i+N-1), \end{aligned} \tag{3.8.12}$$

where

$$\alpha = \frac{x^\ell x^\ell + 4/K}{x^{i+1}} , \quad \beta_r = \frac{x^r}{x^{i+1}} , \quad \gamma = \frac{4\sqrt{-K} x^i + 4 - K x^\ell x^\ell}{-K x^{i+1}} . \tag{3.8.13}$$

Case $K>0$

$$\bar{x}^{i} = \frac{2}{K}\,\frac{\left\{\alpha^2K^2-16K\left(1+\sum_{r=i+2}^{i+N-1}\beta_r^2\right)\right\}^{\frac{1}{2}}}{\alpha K-\left\{\alpha^2K^2-16K\left(1+\sum_{r=i+2}^{i+N-1}\beta_r^2\right)\right\}^{\frac{1}{2}}\sin(\sqrt{K}t-t_0)}$$

$$\bar{x}^{i+1} = \frac{8}{\alpha K-\left\{\alpha^2K^2-16K\left(1+\sum_{r=i+2}^{i+N-1}\beta_r^2\right)\right\}^{\frac{1}{2}}\sin(\sqrt{K}t-t_0)} \qquad (3.8.14)$$

$$\bar{x}^{r} = \beta_r\,\bar{x}^{i+1} \qquad (r=i+2,\dots,i+N-1),$$

where

$$\alpha := \frac{x^\ell x^\ell+4/K}{x^{i+1}}, \quad \beta_r = \frac{x^r}{x^{i+1}},$$

$$e^{i\gamma} = \frac{4\sqrt{K}\,x^i + i(4-Kx^\ell x^\ell)}{\left\{(Kx^\ell x^\ell-4)^2+16K(x^i)^2\right\}^{\frac{1}{2}}} \qquad (3.8.15)$$

Case $K=0$

$$\bar{x}^{i} = x^{i} + t$$
$$\bar{x}^{r} = x^{r} \quad (r\neq i) \qquad (3.8.16)$$

For a proof of the above formulae see [App.8].

The orbits (trajectories) of U^{ij} in any type of CC^N are given by

$$\sigma^{ij}:\ (x^i)^2+(x^j)^2=a^2,\ \bar{x}^r=x^r \qquad (i\neq r, j\neq r) \qquad (3.8.17)$$

The orbits of U^{ij} are circles with centres $(x^1,\dots,x^{i-1},0,x^{i+1},\dots,x^{j-1},0,x^{j+1},\dots,x^N)$ and radii $a=((x^i)^2+(x^j)^2)^{\frac{1}{2}}$. The arbitrary constant a may take any value from 0 to ∞ if $K\geqslant 0$, but $a<2/\sqrt{-K}$ if $K<0$. It is clear that these radii are fictitious in the sense that they do not represent the distance from points on these circles to their centres.

The orbits of U^i $(K \neq 0)$ are given by

$$\sigma^i: \begin{cases} \bar{x}^\ell \bar{x}^\ell + \frac{4}{K} = \alpha \, \bar{x}^{i+1} & (3.8.18a) \\ \bar{x}^r = \beta_r \, \bar{x}^{i+1} \quad (r = i+2, \dots, i+N-1) . & (3.8.18b) \end{cases}$$

From (3.8.18a) we deduce that α can vary from $-\infty$ to $+\infty$ if $K<0$, and

$$|\alpha| \geqslant \frac{4}{\sqrt{K}} \Big(1 + \sum_{r=i+2}^{i+N-1} \beta_r^2 \Big)^{\frac{1}{2}} \tag{3.8.19}$$

if $K>0$. From (3.8.10), (3.8.12), (3.8.14) we deduce the following:

(i) The orbits σ^{ij} of U^{ij} $(i,j=1,\dots,N)$ in any type of CC^N are periodic with period $T = 2\pi$.

(ii) The orbits σ^i of U^i $(i=1,\dots,N)$ when $K>0$ are periodic with period $T = 2\pi/\sqrt{K}$, and they are non-periodic if $K \leqslant 0$.

(iii) If $K<0$, then the orbits of U^i have the property

$$t \to \pm\infty \Rightarrow \bar{x}^i \to \pm 2/\sqrt{-K} \ , \ \bar{x}^j \to 0 \ (j \neq i) . \tag{3.8.20}$$

We may say that the orbits of each U^i meet at infinity since the points $(0,\dots,0,\pm 2/\sqrt{-K},0,\dots)$ lie at infinite distance from any point in a CC^N_- .

(iv) If $K=0$, then as in the case of $K>0$, the orbits of U^i meet at infinity since

$$t \to \pm\infty \Rightarrow \bar{x}^i \to \pm\infty \ , \ \bar{x}^j \to x^j \ (j \neq i) . \tag{3.8.21}$$

To illustrate the above considerations we take the 2-dimensional case. The infinitesimal motions (3.8.8) take the forms

$$\begin{aligned} L_1 &= \{\tfrac{K}{4}[(x^1)^2 - (x^2)^2 + 1\}\, \partial/\partial x^1 + \tfrac{K}{2} x^1 x^2 \, \partial/\partial x^2 , \\ L_2 &= \tfrac{K}{2} x^2 x^1 \, \partial/\partial x^1 + \{\tfrac{K}{4}[(x^2)^2 - (x^1)^2] + 1\}\, \partial/\partial x^2 , \\ L_{12} &= x^1 \, \partial/\partial x^2 - x^2 \, \partial/\partial x^1 . \end{aligned} \tag{3.8.22}$$

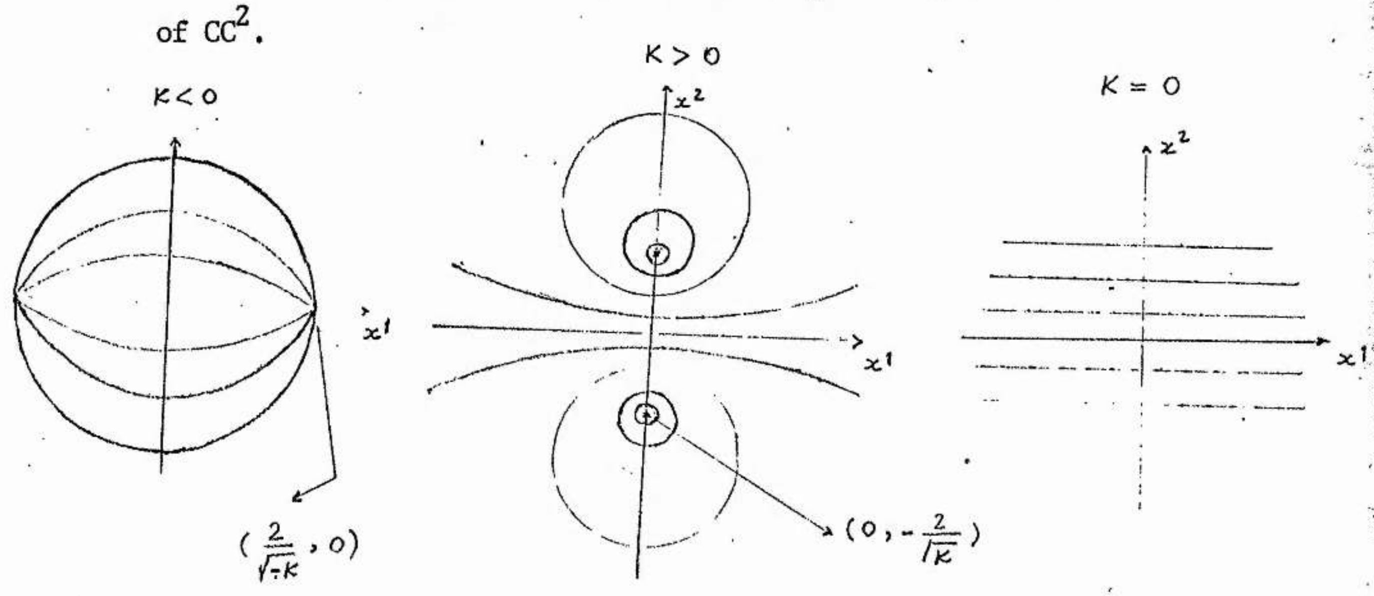
From (3.8.17) and (3.8.18), we find that the orbits of U^1, U^2 and U^{12} are

$$\sigma^1_{\alpha_1} : (\bar{x}^1)^2 + (\bar{x}^2)^2 + 4/K = \alpha_1 x^2 \ , \tag{3.8.23a}$$

$$\sigma^2_{\alpha_2} : (\bar{x}^1)^2 + (\bar{x}^2)^2 + 4/K = \alpha_2 x^2 \ , \tag{3.8.23b}$$

$$\sigma_a^{12}: \ (\bar{x}^1)^2+(\bar{x}^2)^2= a^2 \ , \tag{3.8.23c}$$

respectively. Obviously x^1 and x^2 vary in the range (3.8.1) or in the range (3.8.2) according to K being positive or negative. Equations (3.8.23a) and (3.8.23b) show that $|\alpha_j| \geqslant 4/\sqrt{K}$ $(j=1,2)$ if $K>0$, and α_j $(j=1,2)$ may take any value if $K<0$. Also, these equations show that σ^1 and σ^2 are circles with centres $A_{\alpha_1}(0,\alpha_1/2)$, $A_{\alpha_2}(\alpha_2/2,0)$ and radii $(\alpha_1^2/4-4/K)^{\frac{1}{2}}$ and $(\alpha_2^2/4-4/K)^{\frac{1}{2}}$ respectively. Such centres and radii are fictitious in the sense that even when such orbits are actual circles, namely when $K>0$, the points A_{α_1} and A_{α_2} are not their centres. When $|\alpha_i|= 4/\sqrt{K}$, then the corresponding orbits $\sigma^i_{\alpha_i=4/\sqrt{K}}$ degenerate to the points $(0,\pm 2/\sqrt{K})$ and $(\pm 2/\sqrt{K},0)$. Actually the point $(0,2/\sqrt{K})$ is the centre of $\sigma^1_{\alpha_1}$ with $\alpha_1 \geqslant 4/\sqrt{K}$, while $(0,-2/\sqrt{K})$ is the centre of $\sigma^1_{\alpha_1}$ with $\alpha_1 \leqslant -4/\sqrt{K}$. Similar statements hold for $\sigma^2_{\alpha_2}$ and the points $(\pm 2/\sqrt{K},0)$. The points $(0,2/\sqrt{K})$ and $(2/\sqrt{K},0)$ are opposite poles to $(0,-2/\sqrt{K})$ and $(-2/\sqrt{K},0)$ respectively. If $K<0$, then $\sigma^1_{\alpha_1}$ pass through $(0,\pm 2/\sqrt{-K})$. Obviously $\sigma^1_{\alpha_1}$ and $\sigma^2_{\alpha_2}$ in this case are not actual circles. In either of the cases $K>0$ or $K<0$, σ^{12} are circles with a true common centre $(0,0)$, but of fictitious raduis a. The orbits of U^1, U^2 and U^{12} are depicted in (Fig.(8.2)) for all types of CC^2.



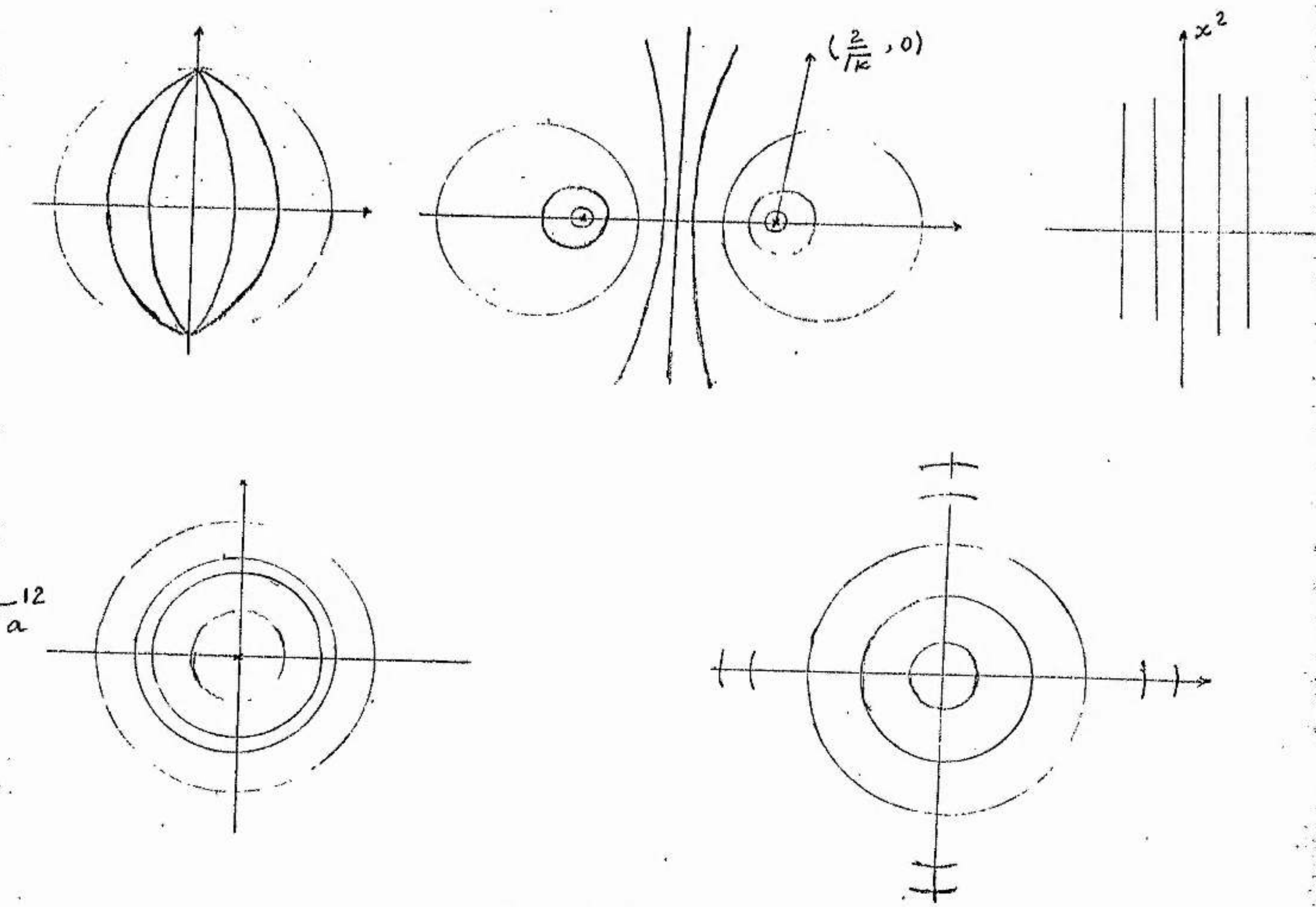


Fig. (8.2)

§(8.4) The Momenta and the Hamiltonian

The quantum momentum observables are

$$\hat{N}_i = -i\hbar\, L_i \quad , \quad \hat{M}_{ij} = -i\hbar\, L_{ij} \;, \tag{3.8.24}$$

where L_i and L_{ij} are given by (3.8.8). It can be verified that

$$\hat{H} = \frac{1}{2m}\Big(\sum_{i=1}^{N} \hat{N}_i^2 + \kappa \sum_{\substack{i=1\\ i<j}}^{N-1} \sum_{j=2}^{N} \hat{M}_{ij}^2\Big) \qquad (i<j) \tag{3.8.25}$$

and explicitly

$$\hat{H} = -\frac{\hbar^2}{2m}\Big\{(1+\frac{\kappa}{4}x^i x^i)^2 \Big(\frac{\partial}{\partial x^\ell}\frac{\partial}{\partial x^\ell}\Big) + (2-N)\,\frac{\kappa}{2}(1+\frac{\kappa}{4}x^i x^i)\Big(x^\ell \frac{\partial}{\partial x^\ell}\Big)\Big\} \tag{3.8.26}$$

The classical momenta are the following functions on T^*CC^N :

$$N_i = \Big\{\frac{\kappa}{4}\Big[(x^i)^2 - \sum_{\substack{j=1\\ j\neq i}}^{N} (x^j)^2\Big] + 1\Big\} p_i + \frac{\kappa}{4}\, x^i \sum_{\substack{j=1\\ j\neq i}}^{N} x^j p_j \;, \tag{3.8.27a}$$

$$M_{ij} = x^i p_j - x^j p_i \qquad (i<j) \,, \tag{3.8.27b}$$

where no summation is implied over i in (3.8.27a). It can be verified that the classical Hamiltonian is related to the classical momenta by the relation

$$H = \frac{1}{2m}\Big(\sum_{i=1}^{N} N_i^2 + K\sum_{\substack{i=1\\ i<j}}^{N-1}\sum_{j=2}^{N} M_{ij}^2\Big), \tag{3.8.28}$$

and explicitly

$$H = \frac{1}{2m}\, g^{ij} p_i p_j = \frac{1}{2m}\big(1+\tfrac{K}{4}x^i x^i\big)^2 (p_\ell p_\ell).$$

From (3.8.9) we see that the classical momenta satisfy the following Poisson bracket relations:

$$\begin{aligned}
\{N_m,N_n\} &= (K\,\delta^r_m\,\delta^s_n - K\,\delta^r_n\,\delta^s_m)\, M_{(rs)}\\
\{N_n,M_{(rs)}\} &= (\delta^m_s\,\delta^k_r - \delta^m_r\,\delta^k_s)\, N_k = -\{M_{(rs)},N_m\},\\
\{M_{(mn)},M_{(rs)}\} &= -(\delta^{nij}_{rms} - \delta^{mij}_{srn} - \delta^{nij}_{smr} + \delta^{nij}_{srm} - \delta^{mij}_{rns} + \delta^{mij}_{rsn})\, M_{(ij)}.
\end{aligned} \tag{3.8.29}$$

When $K \to 0$, we have

$$\hat N_i \to -i\hbar\,\partial/\partial x^i,\quad M_{ij} \to -i\hbar\,(x^i\,\partial/\partial x^j - x^j\,\partial/\partial x^i);$$

$$\hat H \to -\frac{\hbar^2}{2m}\Big(\sum_{i=1}^{N}\hat N_i^2\Big) = -\frac{\hbar^2}{2m}\Big(\frac{\partial}{\partial x^\ell}\frac{\partial}{\partial x^\ell}\Big),$$

which are the quantum momenta and the Hamiltonian in the N-dimensional Euclidean space with rectangular Cartesian coordinates. Similar statements hold in the classical case.

§(8.5) Various Considerations

§(8.5.1) Eigenfunctions and Spectra of the Momentum Observables

Let us adhere again to the convention of §(8.3) according to which we identify q^{s+N} $(s=1,\dots,N)$ by q^s. Let F be an arbitrary function in its arguments, $\lambda\in\mathbb{R}$ and $n\in\mathbb{N}$. Denote the eigenfunctions of $\hat P$

by $\psi_\rho(\hat{P})$ such that $\hat{P}\ \psi_\rho(\hat{P}) = \rho\, \psi_\rho(P)$. Now the eigenfunctions of the momentum operators [App.9] can be written in the following forms :

$\forall$ K

$$\psi_n(\hat{M}_{jk}) = \left\{ \frac{x^j + i x^k}{\sqrt{(x^j)^2 + (x^k)^2}} \right\}^n F((x^j)^2 + (x^k)^2, \underline{x}^r_{jk}) , \tag{3.8.30}$$

where

$$(\underline{x}^r_{jk}) = (x^{j+1}, \ldots, x^{k-1}, x^{k+1}, \ldots, x^{j-1}) ;$$

$K < 0$

$$\psi_\lambda(\hat{N}_k) = \left\{ \frac{4\sqrt{-K}\, x^k + 4 - K x^s x^s}{-K x^{k+1}} \right\}^{i\lambda/\hbar\sqrt{-K}} F\left(\frac{K x^s x^s + 4}{K x^{k+1}}, \frac{\underline{x}^r}{x^{k+1}}\right), \tag{3.8.31a}$$

$K > 0$

$$\psi_n(\hat{N}_k) = \left\{ \frac{4\sqrt{K}\, x^k + i(K x^s x^s - 4)}{\sqrt{16K (x^k)^2 + (K x^s x^s - 4)^2}} \right\}^n F\left(\frac{K x^s x^s + 4}{K x^{k+1}}, \frac{\underline{x}^r}{x^{k+1}}\right), \tag{3.8.31b}$$

where

$$\left(\frac{\underline{x}^r}{x^{k+1}}\right) = \left(\frac{x^{k+2}}{x^{k+1}}, \ldots, \frac{x^{k+N-1}}{x^{k+1}}\right) ,$$

$K = 0$

$$\psi_\lambda(\hat{N}_k) = e^{i\lambda x^k/\hbar}\, \phi(\underline{x}^r_k) , \tag{3.8.31c}$$

where

$$(\underline{x}^r_k) = (x^{k+1}, \ldots, x^{k+N-1}) .$$

The spectra of these momentum operators are given by

$$\left. \begin{aligned} S(\hat{N}_i) &= \sqrt{K}\, \hbar\, \mathbb{N} \text{ (gm-cm/sec)} \\ S(\hat{M}_{ij}) &= \hbar\, \mathbb{N} \text{ (gm-cm}^2\text{/sec)} \end{aligned} \right\} \quad \text{if } K > 0 , \tag{3.8.32a}$$

$$\left. \begin{aligned} S(\hat{N}_i) &= \mathbb{R} \text{ (gm-cm/sec)} \\ S(\hat{M}_{ij}) &= \hbar\, \mathbb{N} \text{ (gm-cm}^2\text{/sec)} \end{aligned} \right\} \quad \text{if } K \leqslant 0 . \tag{3.8.32b}$$

§(8.5.2) Physical Distinction of CC^N of Different Curvatures

Consider the coordinate system $(x^1, \ldots, x^N)$ in a space V of constant curvature $K \neq 0$. Define a new coordinate system in V by

$$y^i = \tfrac{1}{2}\sqrt{eK}\, x^i \quad (i=1, \ldots, N; eK = |K|). \tag{3.8.33}$$

The metric (3.4.2) can be written in terms of $(y^1, \ldots, y^N)$ as

$$ds^2 = (4/\sqrt{eK})\ dy^i\ dy^i/(1+ey^iy^i)^2, \tag{3.8.34}$$

$(|y^i| < 1$ if $K<0$, $-\infty < y^i < +\infty$ if $K \geqslant 0)$.

Let V_1 and V_2 be two CC^N of the same type with curvatures K_1 and K_2 respectively. Let m_1 and m_2 denote arbitrary points in V_1 and V_2 respectively. Now, a homeomorphism A between V_1 and V_2 is defined by

$$m_1 \leftrightarrow m_2 \quad \text{if} \quad \underline{y}(m_1) = \underline{y}(m_2)$$

when K_1 and K_2 are negative, and by

$m_1 \longleftrightarrow m_2$ if $\underline{y}(m_1) = \underline{y}(m_2)$

$O'_1 \longleftrightarrow O'_2$ (O'_i is the point in V_i which is not coverable by the chart $\underline{y}$)

when K_1 and K_2 are positive. The rest of the argument is similar to that of (5.4.2). From (3.8.34) we have

$$K_1\ ds_1^2 = K_2\ ds_2^2 = 4e\ dy^i\ dy^i/(1+ey^iy^i)^2.$$

The map A produces a vector space isomorphism between $L^2(V_1)$ and $L^2(V_2)$. If $(\ ,\ )_i$ denotes the inner product in the Hilbert space $L^2(V_i)$, then

$$(\ ,\ )_2 = (K_1/K_2)^{\frac{1}{2}N}(\ ,\ )_1 .$$

Thus the Hilbert spaces $L^2(V_1)$ and $L^2(V_2)$ are essentially the same. Now, we can show that

$$\langle \hat{H}_1 \rangle / K_1 = \langle \hat{H}_2 \rangle / K_2,\ \langle (N_i)_1 \rangle / \sqrt{eK_1} = \langle (N_i)_2 \rangle / \sqrt{eK} \tag{3.8.35}$$

$$(i = 1, \ldots, N).$$

The relations (3.8.32) show that CC^N of different types are distinguishable. While a CC^N is characterized by the discreteness of the spectra of its momenta, a CC^N is singled out by the simultaneous measurability of the momenta $\hat{N}_i$. Also, by (3.8.35), CC^N of the same type but of different curvatures are quantum-mechanically distinguishable.

§(8.5.3) Unified Treatments of CC^2 and CC^3.

We have discussed in §5 and §6 the problem of quantization in CC^2 and CC^3 considering three separate cases according to the sign of K. However, using the Riemannian forms of CC^2 and CC^3, we can give unified treatments of CC^2 and CC^3.

For CC^2, the momenta $\hat{P}_\mu = -ihL_\mu$, where L_μ are given by (3.8.22) satisfy the commutation relations (3.5.19) and the relation (3.5.20) with

$$\hat{H} = -\frac{\hbar^2}{2m}\left(1+\frac{K}{4}x^i x^i\right)^2 \frac{\partial}{\partial x^\ell}\frac{\partial}{\partial x^\ell}\,. \tag{3.8.36}$$

The present infinitesimal motions are just those of §5 but expressed in terms of a different coordinate system.

For CC^3, we may set $\hat{N}_i = -ihL_i$ and $\hat{M}_i = -ihL_{i+3}$ $(i = 1,2,3)$, when L_μ $(\mu = 1,2,\ldots,6)$ are given by (3.4.7). These momenta satisfy the commutation relations (3.6.38) and the relation (3.6.36) with

$$\hat{H} = \frac{-\hbar^2}{2m}\left\{\left(1+\frac{K}{4}x^i x^i\right)^2\frac{\partial}{\partial x^\ell}\frac{\partial}{\partial x^\ell} - \frac{K}{2}\left(1+\frac{K}{4}x^i x^i\right)x^\ell\frac{\partial}{\partial x^\ell}\right\}. \tag{3.8.37}$$

§9. Lie Algebra and Quantization of the Momenta and the Hamiltonian

§(9.1) Preliminaries

Let U be an OPG of M whose infinitesimal generator is $L = \xi^i\, \partial/\partial x^i$. U induces an OPG of canonical transformations of T*M whose fundamental invariant is $\xi^i p_i$. Also U induces an OPG of unitary transformations of $L^2(M)$ which is generated by a skew-adjoint operator $\frac{i}{h}\hat{P}$, where $\hat{P}\psi = -ih(L + \frac{1}{2}\mathrm{div}L)\psi$, $\forall\ \psi \in D_p \subset C^1(M) \subset L^2(M)$.

Let A, P_C and P_Q denote the sets of infinitesimal motions, classical momenta and quantal momenta in M. Mackey's postulate for quantizing a classical momentum can be rephrased as follows: Quantization of a classical momentum is effected by a map $^\wedge = -ih^\vee$ from P_C to P_Q. This map is defined by

$$\wedge : P_C \to P_Q,\ P = \xi^i p_i \to \hat{P} = -i\hbar\left(\xi^i \frac{\partial}{\partial x^i} + \frac{1}{2}\, div\left(\xi^i \frac{\partial}{\partial x^i}\right)\right). \quad (3.9.1a)$$

The map $^\vee$ takes every classical momentum $P \in P_C$ to a skew-adjoint operator $\frac{i}{h}\hat{P} \in \frac{i}{h} P_Q$, i.e.,

$$\vee : P_C \to \frac{i}{\hbar} P_Q,\ P = \xi^i p_i \to \frac{i}{\hbar}\hat{P} = \left(L + \frac{1}{2}\, div L\right) \quad (= \check{P}). \quad (3.9.1b)$$

Here $\frac{i}{h} P_Q = \{\frac{i}{h}\hat{P} \mid \hat{P} \in P_Q\}$.

If $L_1 = \xi^i\, \partial/\partial x^i$ and $L_2 = \eta^i\, \partial/\partial x^i$ are complete vector fields in M, then their commutator is

$$[L_1, L_2] = L_1L_2 - L_2L_1 = (\xi^i \eta^j_{,i} - \eta^i \xi^j_{,i})\, \partial/\partial x^j. \quad (3.9.2a)$$

Let P_1, P_2 and $\hat{P}_1$, $\hat{P}_2$ be respectively, the classical and quantum momenta corresponding to L_1 and L_2. We define on P_C a new Poisson bracket $\{\ ,\ \}_-$ by

$$\{P_1, P_2\}_- = -\{P_1, P_2\} = (\xi^i \eta^j_{,i} - \eta^i \xi^j_{,i})\, p_j. \quad (3.9.3)$$

This new Poisson bracket will be referred to as Poisson bracket. It is clear that $\{\ ,\ \}_-$ is anti-symmetric in its arguments and it satisfies the Jacobi identity. The commutator bracket

$$[\tfrac{i}{\hbar}\hat{P}_1, \tfrac{i}{\hbar}\hat{P}_2] = (L_1+\tfrac{1}{2}\,div\,L_1)(L_2+\tfrac{1}{2}\,div\,L_2)-(L_2+\tfrac{1}{2}\,div\,L_2)(L_1+\tfrac{1}{2}\,div\,L_1) \quad (3.9.2b)$$

in $\frac{i}{h}P_Q$ is also anti-symmetric and since

$$[L_1+\tfrac{1}{2}\,div\,L_1, L_2+\tfrac{1}{2}\,div\,L_2] = [L_1, L_2] + \tfrac{1}{2}\,div\,[L_1, L_2],$$

the Jacobi identity is readily verified.

It is clear that the map $\vee$ satisfies the properties

$$\begin{aligned} \widecheck{\alpha P_1 + \beta P_2} &= \alpha \check{P}_1 + \beta \check{P}_2 \quad (\alpha, \beta \in \mathbb{R}), \\ \widecheck{\{P_1, P_2\}_-} &= [\check{P}_1, \check{P}_2]\ . \end{aligned} \quad (3.9.4)$$

§(9.2) A Critique of the Lie Algebraic Approach

Among quantization schemes, there is a well known one which is based on Lie algebra [44]. Such a scheme is supposed to be applicable to a reasonably large set of classical observables. The classical momenta P_C is a subset of such. Let us confine our attention to P_C and investigate how a classical momentum can be quantized according to this scheme. The basic lines of this scheme are the following:

(i) P_C form a Lie algebra with respect to the Poisson bracket (3.9.3).

(ii) $\frac{i}{h}P_Q$ form a Lie algebra with respect to the commutator bracket (3.9.2b).

(iii) The map (3.9.1b) defines a representation of the Lie algebra P_C by a Lie algebra $\frac{i}{h}P_Q$ of skew-adjoint operators. Therefore, under quantization Poisson bracket goes over to a commutator bracket.

The considerations of §(9.1) give the impression that everything goes well with the scheme. However, this scheme is not quite correct. Even, if L_1 and L_2 are complete vector fields, it is not necessarily true that $[L_1, L_2]$ is complete [33]. Thus neither is P_C closed under Poisson bracket nor is $\frac{i}{h} P_Q$ closed under the commutator bracket. In other words, $\{P_1, P_2\}_-$ is not a classical momentum since the corresponding vector field $[L_1, L_2]$ is not guaranteed to generate an OPG of M. The incompleteness of this vector field leaves the possibility open for the non-essential self-adjointness of $[\hat{P}_1, \hat{P}_2]$. The exception to this is when the manifold M is compact. Every C^∞ vector field on a compact manifold is complete [33]. As a result a Lie algebra quantization scheme may be carried through. But this is not of fundamental significance since most configuration manifolds of interest including spaces of zero or negative constant curvature are non-compact.

An example of two complete vector fields whose commutator is not complete is the following [33, p.139]:

$$L_1 = x^2 \partial/\partial x^1, \quad L_2 = \tfrac{1}{2}(x^1)^2 \partial/\partial x^2.$$

Here the manifold M is the Euclidean space E^2.

In [44], Hermann states that classical observables of the form $\xi^i(\underline{x})\, p_i$ form a Lie algebra under Poisson brackets and that the assignment

$$\xi^i(\underline{x})\, p_i \rightarrow \xi^i \partial/\partial x^i + \tfrac{1}{2}(\partial \xi^i/\partial x^i) \tag{3.9.5}$$

defines a representation of this Lie algebra by a Lie algebra of skew-adjoint operators. Hermann considers Euclidean spaces and the term $\frac{1}{2}(d\xi^i/\partial x^i)$ in (3.9.5) is just $\frac{1}{2}\mathrm{div}(\xi^i \partial/\partial x^i)$. In view of the above discussion, Hermann's theorem is not quite correct. Neither P_C nor $\frac{i}{h} P_Q$ form a Lie algebra.

These latter considerations show that the Lie algebra quantization scheme cannot be taken as a general quantization scheme.

Let $P_c \subset P_C$ be the set of metric-preserving classical momenta (the set of classical momenta which arise from motions of M), and let P_q be the corresponding set of quantum momenta. We shall show soon that P_c and $\frac{i}{h} P_q$ form Lie algebras. We will refer to these Lie algebras as the Lie algebras of classical and quantum momenta respectively. Also we shall show that the map $^\vee$ is an isomorphism between P_c and $\frac{i}{h} P_q$.

§(9.3) The Lie Algebra of Infinitesimal Motions of CC^N and its Corresponding Classical and Quantum Lie Algebras

Let M be a CC^N. M admits a group of motions G depending on N(N+1)/2 parameters $t_1, t_2, \ldots, t_{N(N+1)/2}$ [34]. The infinitesimal generators of G form a set of N(N+1)/2 independent Killing vectors. Different parametrization of G leads to a new set of independent Killing vectors. Every Killing vector generates an OPG of M which is a subgroup of G.

The infinitesimal motion of M form a Lie algebra A [28]. It is obvious that the vector space requirements are satisfied. Moreover, if L_1 and L_2 are infinitesimal motions corresponding to the motions U_1 and U_2 respectively, then $[L_1, L_2]$ is an infinitesimal motion corresponding to the motion $U_1^{-1}U_2^{-1}U_1U_2$ [41]. It follows that $\frac{i}{h} P_q$ is a Lie algebra. The set of classical momenta P_c could be made into a Lie algebra as well. In addition to the operations of summation and multiplication by a real number which this set of functions on T*M possesses, we define a Lie bracket operation $\{\ ,\ \}_-$ on P_c. It is clear that P_c is closed under $\{\ ,\ \}_-$ and that the Lie algebra requirements are satisfied.

The map $^\vee$ in this case takes the form

$$P = \xi^i p_i \rightarrow \frac{i}{\hbar}\hat{P} = \xi^i \partial/\partial x^i. \tag{3.9.6}$$

This map is a Lie algebra isomorphism [39,41] between P_c and P_q. For it is obvious that $^\vee$ is a vector space isomorphism and that

$$\{P_1, P_2\}_-^\vee = [\check{P}_1, \check{P}_2] \qquad \forall\, P_1, P_2 \in P_c$$

§(9.4) The Hamiltonian and the Casimir Operator C_2 in CC^N

Consider the Lie algebra A of infinitesimal motions of CC^N. Let L_μ ($\mu = 1,\ldots,N(N+1)/2$) be a basis of this Lie algebra. We define on A a metric tensor (a Killing form) [39,41]

$$G_{\mu\nu} = C^{\beta}_{\mu\alpha} C^{\alpha}_{\nu\beta}, \tag{3.9.7}$$

where $C^{\sigma}_{\mu\nu}(\mu,\nu,\sigma = 1,\ldots,N(N+1)/2)$ are the structure constants of the Lie algebra A defined by $[L_\mu, L_\nu] = C^{\sigma}_{\mu\nu} L_\sigma$. If the set (3.8.8) of infinitesimal motions is chosen as a basis of A and the ordering conventions of §(8.2) are adopted, then the structure constants are given by

$$\begin{aligned}
C^{(rs)}_{mn} &= -K\delta^r_m\delta^s_n + K\delta^r_n\delta^s_m,\\
C^{k}_{m(rs)} &= \delta^m_r\delta^k_s - \delta^m_s\delta^k_r = -C^k_{(rs)m},\\
C^{(ij)}_{(mn)(rs)} &= \delta^{nij}_{rms} - \delta^{mij}_{srn} - \delta^{nij}_{smr}\\
&\quad + \delta^{nij}_{srm} - \delta^{mij}_{rns} + \delta^{mij}_{rsn},\\
C^k_{mn} &= 0, \quad C^k_{(mn)(rs)} = 0,\\
C^{(ij)}_{m(rs)} &= C^{(ij)}_{(rs)m} = 0.
\end{aligned} \tag{3.9.8}$$

From (3.9.7) and (3.9.8) we have

$$G_{mn} = c^{k}_{m\ell}\, c^{\ell}_{nk} + c^{(ij)}_{m\ell}\, c^{\ell}_{n(ij)} + c^{k}_{m(ks)}\, c^{(rs)}_{nk} + c^{(ij)}_{m(rs)}\, c^{(rs)}_{n(ij)} = -2K(N-1)\,\delta^{m}_{n}\,,$$

$$\begin{aligned} G_{(mn)(rs)} &= c^{\ell}_{(mn)k}\, c^{k}_{(rs)\ell} + c^{(rs)}_{(mn)k}\, c^{k}_{(rs)} + c^{\ell}_{(mn)(ij)}\, c^{(ij)}_{(rs)\ell} + c^{(ij)}_{(mn)(\ell k)}\, c^{(\ell k)}_{(rs)(ij)} \\ &= -2(N-1)\,\delta^{m}_{r}\,\delta^{n}_{s}\,, \end{aligned}$$

$$G_{m(ij)} = 0\,.$$

Therefore

$$G_{\mu\nu} = -2(N-1) \begin{array}{c} \overset{\longleftarrow\, N \,\longrightarrow}{} \\ \left[\begin{array}{ccc|ccc} K & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & K & & & \\ \hline & & & 1 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & 1 \end{array}\right] \end{array} \updownarrow N(N+1)/2 \tag{3.9.9}$$

This equation shows that $\det|G_{\mu\nu}| \neq 0$ if $K \neq 0$*. By Cartan's criterion A is semi-simple if $K \neq 0$. For a semi-simple Lie algebra the second order Casimir operator** C_2 is defined by [39,41]

$$C_2 = G^{\mu\nu} L_\mu L_\nu\,,$$

where $G_{\mu\nu}\, G^{\mu\sigma} = \delta^{\sigma}_{\nu}$. By (3.9.9)

$$C_2 = \frac{-1}{2(N-1)} \left\{ \frac{1}{K} \sum_{i=1}^{N} L_i^2 + \sum_{(mn)} L^2_{(mn)} \right\}. \tag{3.9.10}$$

---

*Since all CC^1 are Euclidean, the condition $K \neq 0$ implies that $N \neq 1$.

**A Casimir operator of a Lie algebra A is a non-linear function of the basis elements which commutes with all elements of A. Casimir operators can be chosen to be homogeneous polynomials in the basis elements [45]. The degree of the polynomial is by definition the order of the Casimir operator. For subjects related to Casimir operators we refer to [39,41,43,45].

Comparing this relation with (3.8.25) and (3.8.28), we find that

$$H = (1/2m)\, g^{ij} p_i p_j = (1/m)\, K(1-N) G^{\mu\nu} P_\mu P_\nu \tag{3.9.11}$$

and

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 = (1/m)\, K(1-N)\, G^{\mu\nu}\, \hat{P}_\mu \hat{P}_\nu \; . \tag{3.9.12}$$

Denote $G^{\mu\nu} P_\mu P_\nu$ and $G^{\mu\nu} \hat{P}_\mu \hat{P}_\nu$ by C_2^c and C_2^q respectively. Then

$$\{ C_2^c , P \} = 0 \quad \forall P \in P_c \text{ and } [C_2^q , \hat{P}] = 0 \quad \forall \hat{P} \in P_q \, .$$

We may call C_2^c the Casimir operator of the Lie algebra P_c. Also, we say loosely that P_q is a Lie algebra and C_2^q is its Casimir operator. Now we may write (3.9.12) and (3.9.11) as follows:

$$H = (1/m)\, K(1-N)\, C_2^c = (R/mN)\, C_2^c \, ,^* \tag{3.9.13}$$

$$\hat{H} = (1/m)\, K(1-N)\, C_2^q = (R/mN)\, C_2^q \, .^* \tag{3.9.14}$$

Note that while (3.3.10) is not valid for $K = 0$, the relations (3.9.11)-(3.9.14) are valid for the limiting case $K \to 0$. The relation (3.9.11) expresses H in terms of an arbitrary basis of P_c. Similarly (3.9.12) expresses $\hat{H}$ in terms of an arbitrary basis of P_q. When a basis of P_c is chosen such that the Poisson brackets relations (3.8.29) are satisfied, then (3.8.28) holds. A similar statement holds for the quantum case.

Example 1. The Hyperbolic Plane

Consider the Lie algebra A of infinitesimal motions of a CC^2. In terms of the infinitesimal motions L_μ ($\mu = 1,2,3$) given in §(5.2), the metric tensor of A is $(G_{\mu\nu}) = \mathrm{diag}(-2K, -2K, -2)$. Hence the Casimir operator C_2 is given by

$$C_2 = -\tfrac{1}{2}\{(L_1^2 + L_2^2)/K + L_3^2\} \quad (K \neq 0)$$

Comparing this with (3.5.20) and (3.5.14), we find

$$H = (-1/m)\, K\, C_2^c \, , \quad \hat{H} = (-1/m)\, K\, C_2^q \, .$$

---

* Here R is the curvature invariant.

The case of the hyperbolic plane is obtained through setting $K = -1$ in the above formulae. In terms of the basis (3.7.3), the Hamiltonian takes the form (3.7.6a). We may choose L'_μ ($\mu = 1,2,3$) given by (3.7.10) as a basis of A. The metric tensor in terms of this basis is

$$(G_{\mu\nu}) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix}.$$

It follows that $C_2 = \frac{1}{2}(L_1'^2 - L_2' L_3' - L_3' L_2')$, and hence

$$\hat{H} = (1/2m)(\hat{P}_1'^2 - \hat{P}_2' \hat{P}_3' - \hat{P}_3' \hat{P}_2').$$

Example 2. The Hyperbolic Space

Consider the Lie algebra A of infinitesimal motions of the hyperbolic space. By (3.7.32), a choice of a basis of A could be

$$L_1' = \partial/\partial x\,,\ L_2' = \partial/\partial y\,,\ L_3' = y\partial/\partial x - x\partial/\partial y\,,$$
$$L_4' = x\partial/\partial x + y\partial/\partial y + z\partial/\partial z\,,$$
$$L_5' = \tfrac{1}{2}(x^2 - y^2 - z^2)\partial/\partial x + xy\partial/\partial y + xz\partial/\partial z\,,$$
$$L_6' = xy\partial/\partial x + \tfrac{1}{2}(y^2 - x^2 - z^2)\partial/\partial y + yz\partial/\partial z\,.$$

In terms of this basis the metric tensor of A is given by

$$(G_{\mu\nu}) = \begin{bmatrix} 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the Casimir operator C_2 is given by

$$C_2 = \tfrac{1}{4}(L_4'^2 - L_1' L_5' - L_5' L_1' - L_2' L_6' - L_6' L_2').$$

By (3.9.14) the quantum Hamiltonian has the form $\hat{H} = (-2K/m)\, C_2^q$. Thus

$$\hat{H} = (-K/2m)(\hat{P}_4'^2 - \hat{P}_1'\hat{P}_5' - \hat{P}_5'\hat{P}_1' - \hat{P}_2'\hat{P}_6' - \hat{P}_6'\hat{P}_2')\,.$$

Let

$$C_r^c = a^{\mu_1\cdots\mu_r} P_{\mu_1}\cdots P_{\mu_r} \quad (\mu_1,\ldots,\mu_r = 1,\ldots,N(N+1)/2;\, a^{\mu_1\cdots\mu_r} \in \mathbb{R})$$

be a classical observable. As a consequence of the isomorphism (3.9.6) we find that

$$\{a^{\mu_1\cdots\mu_r} P_{\mu_1}\cdots P_{\mu_r}, P_o\} = 0 \quad (P_o \in P_c)$$

iff its quantum analogue

$$C_r^q = a^{\mu_1\cdots\mu_r}\hat{P}_{\mu_1}\cdots\hat{P}_{\mu_r}$$

commutes with $\hat{P}_o$, i.e. iff

$$[C_r^q, \hat{P}_o] = 0\,.$$

(C_r^c is a homogeneous polynomial in the P's. However, the above statement is true for any polynomial in the P's). If $C_r = G^{\mu_1\ldots\mu_r} L_{\mu_1}\ldots L_{\mu_r}$ is a Casimir operator of order r, then $C_r^c = G^{\mu_1\ldots\mu_r} P_{\mu_1}\ldots P_{\mu_r}$ has a vanishing Poisson bracket with every classical momentum $P \in P_c$. Also $C_r^q = G^{\mu_1\ldots\mu_r}\hat{P}_{\mu_1}\ldots\hat{P}_{\mu_r}$ commutes with all quantum momenta.

APPENDIX 1

The Infinitesiml Motions of CC^N

The Killing equations are

$$\xi^1_{,1} = \cdots = \xi^N_{,N} = \frac{K}{2} x^\ell \xi^\ell / (1 + \frac{K}{4} x^\ell x^\ell), \tag{a1.1}$$

$$\xi^\ell_{,k} + \xi^k_{,\ell} = 0 \quad (k \neq \ell ; k, \ell = 1, \ldots, N) . \tag{a1.2}$$

Let

$$\phi = x^\ell \xi^\ell . \tag{a1.3}$$

Then

$$\phi_{,k} = \xi^k + x^\ell \xi^\ell_{,k} .$$

Multiplying this equation by x^k and summing over k, we get

$$\begin{aligned} x^k \phi_{,k} &= x^k \xi^k + x^k x^\ell \xi^\ell_{,k} \\ &= x^k \xi^k + \sum_{\ell=1}^{N} (x^\ell)^2 \xi^\ell_{,\ell} && \text{(by (a1.2))} \\ &= x^k \xi^k + (x^\ell x^\ell)\{\tfrac{K}{2} x^\ell \xi^\ell / (1 + \tfrac{K}{4} x^\ell x^\ell)\} && \text{(by (a1.1))} \\ &= \phi \{ 1 + \tfrac{K}{2} x^\ell x^\ell / (1 + \tfrac{K}{4} x^\ell x^\ell) \} && \text{(by (a1.3))}. \end{aligned} \tag{a1.4}$$

This is a quasi-linear partial differential equation. Following the standard method for solving such type of equation, we set

$$\frac{dx^1}{x^1} = \cdots = \frac{dx^N}{x^N} = \frac{d\phi}{\phi\{1 + \frac{K}{2} x^\ell x^\ell / (1 + \frac{K}{4} x^\ell x^\ell)\}} . \tag{a1.5}$$

The above system admits the following (N-1) obvious integrals

$$x^r = c_r x^s \qquad (r \neq s ; r = 1, \ldots, N). \tag{a1.6}$$

Substituting for x^r from (a1.6) in

$$\frac{dx^s}{x^s} = \frac{d\phi}{\phi\{1 + \frac{K}{2} x^\ell x^\ell / (1 + \frac{K}{4} x^\ell x^\ell)\}}$$

and integrating, we find

$$\phi = C_{(S)}\, x^{(S)}\left\{1+\frac{K}{4}(x^s)^2\left(1+\sum_{\substack{r=1\\ r\neq s}}^{N} c_r^2\right)\right\}.$$

By (a1.6),

$$\phi = C_{(S)}\, x^{(S)}\left\{1+\frac{K}{4}\,x^\ell x^\ell\right\}.$$

Therefore, the general solution of (a1.5) is

$$\Omega'\left(\phi/\{x^s(1+\frac{K}{4}x^\ell x^\ell)\},\ \underline{x}^r/x^s\right)=0, \tag{a1.7}$$

where Ω' is an arbitrary function in its arguments and $\underline{x}^r/x^s=(x^1/x^s,$ $\ldots, x^{s-1}/x^s, x^{s+1}, \ldots, x^N/x^s)$. Solving (a1.7) for ϕ,

$$\phi = x^s\left(1+\frac{K}{4}x^\ell x^\ell\right)\,\Omega(\underline{x}^r/x^s). \tag{a1.8}$$

The only choice of Ω which allows for a symmetrical dependence of ϕ on x^i $(i=1, \ldots, N)$ is the following:

$$\Omega = \alpha_\ell\, x^\ell/x^s. \tag{a1.9}$$

From (a1.8) and (a1.9)

$$\phi = (\alpha_\ell\, x^\ell)\left(1+\frac{K}{4}x^\ell x^\ell\right). \tag{a1.10}$$

Substituting for ϕ in (a1.1),

$$\xi^1_{,1} = \cdots = \xi^N_{,N} = \frac{K}{2}\alpha_\ell\, x^\ell. \tag{a1.11}$$

(In order to avoid ambiguity, the summation convention over repeated indices is abondoned in the rest of this appendix. From (a1.11)

$$\xi^r = \frac{K}{4}\alpha_r (x^r)^2 + \frac{K}{2}\sum_{\substack{j=1\\ j\neq r}}^{N}\alpha_j x^j x^r + f_r(x^1,\ldots,x^{r-1},x^{r+1},\ldots,x^N) \tag{a1.12}$$

Consider ξ^r and ξ^s. From (a1.12) and (a1.2)

$$\frac{K}{2}\alpha_s\, x^r + \frac{\partial f_r}{\partial x^s} + \frac{K}{2}\alpha_r\, x^s + \frac{\partial f_s}{\partial x^r} = 0$$

$$\Longrightarrow\quad \frac{K}{2}\alpha_s\, x^r + \frac{\partial f_s}{\partial x^r} = \beta_r^s \;,\quad \frac{K}{2}\alpha_r\, x^s + \frac{\partial f_r}{\partial x^s} = \beta_s^r \;\;(= -\beta_r^s)$$

$$\Longrightarrow\quad \begin{cases} f_s = -\frac{K}{4}\alpha_s\,(x^r)^2 + \beta_r^s\, x^r + f_{sr} \\ f_r = -\frac{K}{4}\alpha_r\,(x^s)^2 + \beta_s^r\, x^s + f_{rs}\,, \end{cases}$$

where f_{rs} and f_{sr} are functions which are independent of x^r and x^s. Thus,

$$\xi^r = \frac{K}{4}\alpha_r\left[(x^r)^2 - (x^s)^2\right] + \frac{K}{2}\sum_{\substack{j=1\\ j\neq r}}^{N} \alpha_j\, x^j x^r + \beta_s^r\, x^s + f_{rs}\;,$$

$$\xi^s = \frac{K}{4}\alpha_s\left[(x^s)^2 - (x^r)^2\right] + \frac{K}{2}\sum_{\substack{j=1\\ j\neq s}} \alpha_j\, x^j x^s + \beta_r^s\, x^r + f_{sr}\,.$$

It is straightforward to carry on this analysis considering ξ^r and ξ^s with the rest of components and making use of (a1.2) to find

$$\xi^r = \frac{K}{4}\alpha_r\Big[(x^r)^2 - \sum_{\substack{j=1\\ j\neq r}}^{N}(x^j)^2\Big] + \frac{K}{2}\sum_{\substack{j=1\\ j\neq r}}^{N}\alpha_j\, x^j x^r + \sum_{\substack{j=1\\ j\neq r}}^{N}\beta_j^r\, x^j + \gamma_r\;,$$

$$\xi^s = \frac{K}{4}\alpha_s\Big[(x^s)^2 - \sum_{\substack{j=1\\ j\neq s}}^{N}(x^j)^2\Big] + \frac{K}{2}\sum_{\substack{j=1\\ j\neq s}}^{N}\alpha_j\, x^j x^s + \sum_{\substack{j=1\\ j\neq s}}^{N}\beta_r^j\, x^j + \gamma_s\;,$$

where $\beta_j^r = -\beta_r^j$. By (a1.1) we find that $\alpha_\ell = \gamma_\ell$ and hence

$$\xi^r = \alpha_r\Big\{1 + \frac{K}{4}\Big[(x^r)^2 - \sum_{\substack{j=1\\ j\neq r}}^{N}(x^j)^2\Big]\Big\} + \frac{K}{2}\sum_{\substack{j=1\\ j\neq r}}^{N}\alpha_j\, x^j x^r + \sum_{\substack{j=1\\ j\neq r}}^{N}\beta_j^r\, x^j\;,$$

$$\xi^s = \alpha_s\Big\{1 + \frac{K}{4}\Big[(x^s)^2 - \sum_{\substack{j=1\\ j\neq s}}^{N}(x^j)^2\Big]\Big\} + \frac{K}{2}\sum_{\substack{j=1\\ j\neq s}}^{N}\alpha_j\, x^j x^s + \sum_{\substack{j=1\\ j\neq s}}^{N}\beta_r^j\, x^j\,.$$

A choice of $N(N+1)/2$ independent infinitesimal motions could be

$$L_i = \Big\{1 + \frac{K}{4}\Big[(x^i)^2 + \sum_{\substack{j=1\\ j\neq i}}^{N} (x^j)^2\Big]\Big\}\partial/\partial x^i + \frac{K}{2}x^i \sum_{\substack{j=1\\ j\neq i}}^{N} x^j \partial/\partial x^j \quad (i = 1, \ldots, N),$$

$$L_{rs} = x^r\partial/\partial x^s - x^s\partial/\partial x^r \qquad (r < s;\ r, s = 1, \ldots, N).$$

APPENDIX 2

Proof of Lemma (4.2)

Consider the matrix formed by the components of L_1, $L_{31} = L_5$ and $L_{12} = L_6$. The determinant of the matrix (ξ^i_ν) $(i = 1,2,3;\ \nu = 1,6,5)$ is

$$\begin{vmatrix} \frac{K}{4}[(x^1)^2-(x^2)^2-(x^3)^2]+1 & \frac{K}{2}x^1x^2 & \frac{K}{2}x^1x^3 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{vmatrix} = (x^1)^2\sqrt{G} \ .$$

Similarly, the matrices (ξ^i_μ) $(i = 1,2,3;\ \nu = 2,6,4)$ and (ξ^i_μ) $(i=1,2,3;\ \nu = 3,4,5)$ have determinants $(x^2)^2\sqrt{G}$ and $(x^3)^2\sqrt{G}$ respectively. Thus the matrix (ξ^i_μ) $(i = 1,2,3;\ \mu = 1,\ldots,6)$ is of rank 3 everywhere, possibly apart from the point $x^1 = x^2 = x^3 = 0$. However, the matrix $(\xi^i_\mu(0,0,0))$ $(i = 1,2,3;\ \mu = 1,2,3)$ has unit determinant . Thus (ξ^i_μ) is of rank 3 everywhere. This lemma proves the intuitively obvious fact that the totality of directions defined by the totality of Killing vectors ensuing from any point in the space furnish all possible directions in the space at that point.

APPENDIX 3

Proof of Lemma (4.3)

The equations

$$[b^m \partial/\partial x^m, L_\mu] = 0 \quad (\mu=1, \ldots, 6) \qquad \text{(a3.1)}$$

are satisfied iff the equations

$$b^m_{,j}\, \xi^j_\mu - b^j \xi^m_{\mu,j} = 0 \quad (m,j=1,2,3;\mu=1, \ldots, 6) \qquad \text{(a3.2)}$$

are satisfied. Writing (a3.2) explicitly for L_4, L_5 and L_6, we find

$$\begin{aligned}
&x^2 b^1_{,1} - x^1 b^1_{,2} = b^2, \quad x^3 b^1_{,1} - x^1 b^1_{,3} = b^3, \quad x^3 b^1_{,2} - x^2 b^1_{,3} = 0 \quad ; \\
&x^2 b^2_{,1} - x^1 b^2_{,2} = -b^1, \quad x^3 b^2_{,1} - x^1 b^2_{,3} = 0, \quad x^3 b^2_{,2} - x^2 b^2_{,3} = b^3 \quad ; \qquad \text{(a3,3)} \\
&x^2 b^3_{,1} - x^1 b^3_{,2} = 0, \quad x^3 b^3_{,1} - x^1 b^2_{,3} = -b^1, \quad x^3 b^3_{,2} - x^2 b^3_{,2} = -b^2 \quad .
\end{aligned}$$

It is easy to show that these nine equations imply that

$$b^2 = x^2\, b^1/x^1 \quad , \quad b^3 = x^3\, b^1/x^1, \qquad \text{(a3.4)}$$

Now, setting $Y=Z=b^m \partial/\partial x^m$ in (1.2.13) and making use of (a3.4), we find

$$L(b^m\, b^m/G) = 0 \qquad \text{(a3,5)}$$

This last equation is valid for any Killing vector L. Since the totality of directions at any point of the space are furnished by the directions defined by the infinitesimal motions at that point, we get

$$b^m\, b^m/G = \alpha^2 \quad (\alpha \text{ is a constant }) \qquad \text{(a3.6)}$$

everywhere. By (a3.4) and (a3.6) we have

$$b^m \partial/\partial x^m = \left\{\alpha\, (1+ \tfrac{K}{4}\, x^i x^i)/\sqrt{x^s x^s}\right\}\, x^m \partial/\partial x^m. \qquad \text{(a3.7)}$$

But it can be shown that α in (a3.7) must vanish if II is to be satisfied

Consider for example the single equation $b^2_{,j}\,\xi^j_1 = b^j\,\xi^2_{1,j}$ of the system II. Routine calculations show that $\alpha G = 0$, which implies $\alpha = 0$. It follows from (a3.5) that $b^1 = b^2 = b^3 = 0$.

APPENDIX 4

The Infinitesimal Motions of CC^2 in Geodesic Coordinats

1. The Case of CC^2_- .

From the metric form (3,5,1a) we have the Killing equations

$$\xi^1_{,1} = 0 \tag{a4,1}$$

$$\xi^2_{,2} + \sqrt{-K}\, th\sqrt{-K}\,x\ \xi^1 = 0 \tag{a4,2}$$

$$\xi^1_{,2} + ch^2\sqrt{-K}\,x\ \xi^2_{,1} = 0 \tag{a4,3}$$

Equation (a4.1) shows that ξ^1 is independent of x, and hence $\xi^1 = \xi^1(y)$. On integrating (a4.2) with respect to y, we get

$$\xi^2 = -\sqrt{-K}\ th\sqrt{-K}\,x \int_0^y \xi^1(y)\,dy + f(x). \tag{a4,4}$$

Substituting for ξ^1 and ξ^2 in (a4,3),

$$\frac{d\xi^1(y)}{dy} + K\int_0^y \xi^1(y)\,dy + ch^2\sqrt{-K}\,x\ \frac{df(x)}{dx} = 0.$$

This equation gives

$$\frac{d\xi^1(y)}{dy} + K\int_0^y \xi^1(y)\,dy = -\beta, \tag{a4,5}$$

$$ch^2\sqrt{-K}\,x\ \frac{df(x)}{dx} = \beta, \tag{a4,6}$$

where β is an arbitrary constant. The differentiation of (a4,5) with respect to y yields

$$\frac{d^2\xi^1(y)}{dy^2} + K\,\xi(y) = 0,$$

The general solution of this equation is

$$\xi^1 = \alpha\, ch\sqrt{-K}\,y + B\, sh\sqrt{-K}\,y, \tag{a4.7}$$

where α and B are arbitrary constants, Substituting for ξ^1 from (a4,7) in (a4.5), we get

$$\sqrt{-K}(\alpha\, sh\sqrt{-K}\, y + B\, ch\sqrt{-K}\, y) - \sqrt{-K}\left[\alpha\, sh\sqrt{-K}\, y + B\, ch\sqrt{-K}\, y\right]_0^y = -\beta$$

$$\Longrightarrow \quad B = -\beta/\sqrt{-K} \ ,$$

and hence

$$\xi^1 = \alpha\, ch\sqrt{-K}\, y - \frac{\beta}{\sqrt{-K}}\, sh\sqrt{-K}\, y. \qquad (a4.8)$$

The integration of (a4.6) gives

$$f(x) = \frac{\beta}{\sqrt{-K}}\, th\sqrt{-K}\, x + \gamma \cdot \qquad (a4.9)$$

where γ is an arbitrary constant. From (a4.4), (a4.8), (a4.9) we have

$$\xi^2 = -\sqrt{-K}\, th\sqrt{-K}\, x \int_0^y (\alpha\, ch\sqrt{-K}\, y - \frac{\beta}{\sqrt{-K}}\, sh\sqrt{-K}\, y)\, dy + \frac{\beta}{\sqrt{-K}}\, th\sqrt{-K}\, x + \gamma$$

$$= -th\sqrt{-K}\, x\, (\alpha\, sh\sqrt{-K}\, y - \frac{\beta}{\sqrt{-K}}\, ch\sqrt{-K}\, y) + \gamma. \qquad (a4.10)$$

From (a4.8) and (a4.10) we deduce that any infinitesimal motion of CC^2_- is of the form

$$L = (\alpha\, ch\sqrt{-K}\, y - \frac{\beta}{\sqrt{-K}}\, sh\sqrt{-K}\, y)\partial/\partial x + \{-th\sqrt{-K}\, x(\alpha\, sh\sqrt{-K}\, y - \frac{\beta}{\sqrt{-K}}\, ch\sqrt{-K}\, y) + \gamma\}\partial/\partial y \qquad (a4.11)$$

where α, β and γ are arbitrary constants,

2. The Case of CC^2_+ .

The proof is similar to that of the previous case, The result is

$$L = (\alpha \cos\sqrt{K}\, y + \beta \sin\sqrt{K}\, y)\partial/\partial x + \{\tan\sqrt{K}\, x\, (\alpha \sin\sqrt{K}\, y - \frac{\beta}{\sqrt{K}} \cos\sqrt{K}\, y) + \gamma\}\partial/\partial y \qquad (a4.12)$$

where α, β and γ are arbitrary constants,

3, The Case of CC^2_0 .

This may be regarded as the limiting case as K tends to zero either from above or below. The general expression of an infinitesimal motion of E^2, as could be seen from either (a4.11) or (a4.12), is of the form

$$L = (\alpha - \beta\, y)\partial/\partial x + (\gamma + \beta\, x)\partial/\partial y. \qquad (a4.11)$$

APPENDIX 5

Infinitesimal Motions of the Hyperbolic Space and the Hyperbolic Plane

From (3.7.31), the Killing equations are

$$\xi^1_{,1} = \xi^3/z \;(a), \quad \xi^2_{,2} = \xi^3/z \;(b), \quad \xi^3_{,3} = \xi^3/z \;(c), \tag{a5.1}$$

$$\xi^1_{,2} + \xi^2_{,1} = 0 \;(a), \quad \xi^1_{,2} + \xi^2_{,1} = 0 \;(b), \quad \xi^2_{,3} + \xi^3_{,2} = 0 \;(c). \tag{a5.2}$$

From (a5.1c) we have

$$\xi^3 = z\, f(x,y). \tag{a5.3}$$

From (a5.3), (a5.2b) and (a5.2c),

$$\xi^1 = \int_0^x f(x,z)\, dx + g_1(y,z) \quad , \quad \xi^2 = \int_0^y f(x,y)\, dy + g_2(x,z). \tag{a5.4}$$

From (a5.4), (a5.2b) and (a5.2c),

$$\frac{\partial g_1(y,z)}{\partial z} + z\,\frac{\partial f(x,z)}{\partial x} = 0 \quad , \quad \frac{\partial g_2(x,z)}{\partial z} + z\,\frac{\partial f(x,y)}{\partial y} = 0. \tag{a5.5}$$

Equations (a5.5) show that f is a linear function in x and y, and hence

$$f = \alpha_1 x + \alpha_2 y + \beta. \tag{a5.6}$$

Substituting for f in (a5.5) and integrating, we get

$$g_1 = -\frac{\alpha_1}{2} z^2 + Y(y), \qquad g_2 = -\frac{\alpha_2}{2} z^2 + X(x). \tag{a5.7}$$

By (a5.4), (a5.6) and (a5.7) we have

$$\xi^1 = \frac{\alpha_1}{2}(x^2 - z^2) + \alpha_2 x y + \beta x + Y(y), \quad \xi^2 = \frac{\alpha_2}{2}(y^2 - z^2) + \alpha_1 x y + \beta y + X(x). \tag{a5.8}$$

Substituting for ξ^1 and ξ^2 in (a5.2a),

$$\alpha_2 x + \frac{\partial Y}{\partial y} + \alpha_1 y + \frac{\partial X}{\partial x} = 0$$

$$-\alpha_2 x - \frac{\partial X}{\partial x} = \alpha_1 y + \frac{\partial Y}{\partial y} = k ,$$

and hence

$$X = -\frac{\alpha_2}{2} x^2 - kx + \gamma_1 , \quad Y = -\frac{\alpha_1}{2} y^2 + ky + \gamma_2 . \tag{a5.9}$$

Thus infinitesimal motions of the hyperbolic space are of the form

$$\begin{aligned} L &= \left\{\frac{\alpha_1}{2}(x^2 - y^2 - z^2) + \alpha_2 xy + \beta x + ky + \gamma_1\right\} \partial/\partial x \\ &+ \left\{\frac{\alpha_2}{2}(y^2 - x^2 - z^2) + \alpha_1 xy + \beta y - kx + \gamma_2\right\} \partial/\partial y + z(\alpha_1 x + \alpha_2 y + \beta)\partial/\partial z \end{aligned} \tag{a5.10}$$

We note that we can obtain the infinitesimal motions of the hyperbolic plane through setting $y=0$, $\partial/\partial y = 0$ in (a5.10). The result is

$$L = \left\{\frac{\alpha_1}{2}(x^2 - z^2) + \beta x + \gamma_1\right\} \partial/\partial x + z(\alpha_1 x + \beta)\partial/\partial z$$

Here the coordinate system employed in the hyperbolic plane is (x,z).

APPENDIX 6

Motions of the Hyperbolic Space.

The motion U generated by the infinitesimal motion $L = \xi^1\partial/\partial x + \xi^2\partial/\partial y + \xi^3\partial/\partial z$ is the solution of the system of ordinary differential equations

$$\begin{cases} \dfrac{d\bar{x}}{dt} = \xi^1(\bar{x},\bar{y},\bar{z}) \\ \dfrac{d\bar{y}}{dt} = \xi^2(\bar{x},\bar{y},\bar{z}) \\ \dfrac{d\bar{z}}{dt} = \xi^3(\bar{x},\bar{y},\bar{z}) \end{cases} \tag{a6.1}$$

with the initial conditions $(x(0),y(0),z(0)) = (x,y,z)$.

(i) The system of differential equations corresponding to L_1 may be written as

$$\frac{dx}{xy} = \frac{dy}{\frac{1}{2}(y^2-x^2-z^2-1)} = \frac{dz}{yz} = dt \tag{a6.2}$$

$$\Longrightarrow \begin{cases} \dfrac{dx}{x} = \dfrac{dz}{z} & \text{(a6.3)} \\ \dfrac{dx}{x} = \dfrac{y\,dy}{\frac{1}{2}(y^2-x^2-z^2-1)} = \dfrac{dz}{z}. & \text{(a6.4)} \end{cases}$$

From (a6.4) we have

$$\frac{dz}{z} = \frac{x\,dx+y\,dy+z\,dz}{\frac{1}{2}(x^2+y^2+z^2+1)} \tag{a6.5}$$

On integrating (a6.4) and (a6.5), we get

$$x = \alpha z \quad , \tag{a6.6}$$

$$x^2+y^2+z^2-1 = \beta z \;, \tag{a6.7}$$

where α and β are arbitrary constants. These two equations give the integral curves of L_1 (the orbits of U^1). From (a6.6) and (a6.7) we have

$$y^2 = -(1+\alpha^2)z^2 + \beta z + 1 . \tag{a6.8}$$

Substituting for y from (a6.8) in the equation $dt = dz/yz$,

$$dt = \frac{dz}{z\sqrt{-(1+\alpha^2)z^2+\beta z+1}} .$$

Setting $z=1/w$ and integrating,

$$-t = \int \frac{dw}{\sqrt{w^2+\beta w-(1+\alpha^2)}} = \ln\left|\frac{1}{\gamma}\left[2\sqrt{w^2+\beta w-(1+\alpha^2)}+2w+\beta\right]\right| \cdot [42] \tag{a6.9}$$

Substituting for w by $1/z$ and for β and α from (a6.6) and (a6.7) in this last equation we then find

$$\gamma e^{-t} = [x^2+(y+1)^2+z^2]/z . \tag{a6.10}$$

Solving for x,y and z from (a6.6), (a6.7) and (a6.10),

$$\bar{x}(t) = \frac{4\alpha\gamma e^{-t}}{(\gamma e^{-t}-\beta)^2+4(1+\alpha^2)} ,$$

$$\bar{y}(t) = \frac{2\gamma e^{-t}(\gamma e^{-t}-\beta)}{(\gamma e^{-t}-\beta)^2+4(1+\alpha^2)} - 1 , \tag{a6.11}$$

$$\bar{z}(t) = \frac{4\gamma e^{-t}}{(\gamma e^{-t}-\beta)^2+4(1+\alpha^2)} ,$$

where

$$\alpha = x/z , \quad \beta = (x^2+y^2+z^2-1)/z \; ; \quad \gamma = [x^2+(y+1)^2+z^2]/z . \tag{a6.12}$$

The method of deriving the OPG U^2 generated by L_2 and its orbits is similar to (i). The OPG's generated by L_3 and L_6 are easy to derive.

(iv) The system of differential equations associated with L_4 is

$$\frac{dx}{\frac{1}{2}(x^2-y^2-z^2+1)} = \frac{dy}{xy} = \frac{dz}{xz} = dt\,. \tag{a6.13}$$

$$\Rightarrow \begin{cases} \dfrac{dy}{y} = \dfrac{dz}{z}\,, & \text{(a6.14)} \\[2ex] \dfrac{dy}{y} = \dfrac{dz}{z} = \dfrac{x\,dx}{\frac{1}{2}(x^2-y^2-z^2+1)}\,. & \text{(a6.15)} \end{cases}$$

From (a6.15) we have

$$\frac{dz}{z} = \frac{x\,dx + y\,dy + z\,dz}{\frac{1}{2}(x^2+y^2+z^2+1)} \tag{a6.16}$$

From (a6.14) and (a6.16) we have two integrals of the system (a6.13),

$$y = \alpha z\,, \tag{a6.17}$$

$$x^2+y^2+z^2+1 = \beta z\,. \tag{a6.18}$$

The orbits of U^4 are given by these two equations, and from them we find

$$x^2 = -(1+\alpha^2)z^2 + \beta z - 1\,. \tag{a6.19}$$

Substituting for x from (a6.19) in $dt = dz/xz$,

$$dt = \frac{dz}{z\sqrt{-(1+\alpha^2)z^2+\beta z-1}}$$

Setting $z=1/w$ and integrating,

$$t = -\int \frac{dw}{\sqrt{-w^2+\beta w-(1+\alpha^2)}} \tag{a6.20}$$

$$= \gamma + \arcsin\frac{-2w+\beta}{\sqrt{\beta^2-4(1+\alpha^2)}} \tag{a6.21}$$

Substituting for w by $1/z$ and for α and β from (a6.17) and (a6.18), we get

$$\sin(t-\gamma) = \frac{x^2+y^2+z^2-1}{\sqrt{(x^2+y^2+z^2+1)^2+4x^2}} \quad . \tag{a6.22}$$

This last expression shows that the condition under which we can put the result of (a6.20) in the form (a6.21) is satisfied [42]. From (a6.17), (a6.18) and (a6.22), we have

$$\sin(t-\gamma) = \frac{\beta z - 2}{\sqrt{(\beta z-2)^2 - 4(1+\alpha^2)z^2 + 4\beta z - 4}} \quad ,$$

and hence

$$\bar{z}(t) = \frac{2}{\beta - \sqrt{\beta^2 - 4(1+\alpha^2)}\,\sin(t-\gamma)} \quad . \tag{a6.23}$$

From (a6.23) and (a6.17),

$$\bar{y}(t) = \frac{2\alpha}{\beta - \sqrt{\beta^2 - 4(1+\alpha^2)}\,\sin(t-\gamma)} \quad . \tag{a6.24}$$

From (a6.19) and (a6.24),

$$\bar{x}(t) = \frac{\sqrt{\beta^2 - 4(1+\alpha^2)}\,\cos(t-\gamma)}{\beta - \sqrt{\beta^2 - 4(1+\alpha^2)}\,\sin(t-\gamma)} \quad .$$

The constants α and β are given by (a6.17) and (a6.18), while γ is given by

$$\sin\gamma = \frac{1-(x^2+y^2+z^2)}{\sqrt{(x^2+y^2+z^2-1)^2+4x^2}}, \quad \cos\gamma = \frac{2x}{\sqrt{(x^2+y^2+z^2-1)^2+4x^2}} \quad .$$

From (a6.19) we have

(v) The derivation of U^5 is similar to that of U^4.

APPENDIX 7

The Polar Coordinate System in CC^N.

A system of geodesic polar coordinates can be introduced in CC^N. We start with N mutually orthogonal geodesics $OG_1, \ldots, OG_N$ passing through some point O. If m is any point in the space, then m is specified by its distance r from O in the direction of the geodesic Om. But the geodesic Om is determined by its direction at O, and hence by the angle θ_1 which it makes with OG_N, and by the direction of its prjection Om_1 in the (N-1)-dimensional vector space spanned by the tangent vectors $v_1, \ldots, v_{N-1}$ to the geodesics $OG_1, \ldots, OG_{N-1}$ at O. This projection in turn is determined by the angle θ_2 which it makes with OG_{N-1} and by the direction of its projection on the (n-2)-dimensional vector space spanned by $v_1, \ldots, v_{N-2}$, and so on. Consequently m is determined by $(r, \theta_1, \ldots, \theta_{N-2}, \phi)$, where

$$0 \leqslant \theta_1, \cdots, \theta_{N-2} < \pi, \quad 0 \leqslant \phi < 2\pi, \qquad 0 \leqslant r < \pi/\sqrt{K} \text{ if } K>0 \text{ and } r \geqslant 0 \text{ if } K \leqslant 0. \tag{a7.1}$$

Following a similar method to that given by Synge and Schild [35] in deriving the metric form of CC^3 in terms of geodesic polar coordinates, we find that the metric form of CC^N in terms of $(r, \theta_1, \ldots, \theta_{N-1}, \phi)$ is given by

$$ds^2 = dr^2 + \frac{1}{K}\sin^2\sqrt{K}r\,(d\theta_1^2 + \sin^2\theta_1\, d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2\, d\theta_3^2 + \cdots + \sin^2\theta_1 \sin^2\theta_2 \cdots \sin^2\theta_{N-2}\, d\phi^2) \quad \text{if } K<0. \tag{a7.2}$$

and

$$ds^2 = dr^2 - \frac{1}{K}\,\mathrm{sh}^2\sqrt{-K}\,r\,(d\theta_1^2 + \sin^2\theta_1\, d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2 + \cdots + \sin^2\theta_1 \sin^2\theta_2 \cdots \sin^2\theta_{N-2}\, d\phi^2) \quad \text{if } K>0 \tag{a7.3}$$

It can be verified that the coordinate transformation between $(r, \theta_1, \ldots, \theta_{N-2}, \phi)$ and $(x^1, \ldots, x^N)$ is given by

$$
\begin{aligned}
x^1 &= \frac{2}{\sqrt{K}} \tan\frac{\sqrt{K}}{2} r \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{N-3} \sin\theta_{N-2} \sin\phi , \\
x^2 &= \frac{2}{\sqrt{K}} \tan\frac{\sqrt{K}}{2} r \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{N-3} \sin\theta_{N-2} \cos\phi , \\
x^3 &= \frac{2}{\sqrt{K}} \tan\frac{\sqrt{K}}{2} r \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{N-3} \cos\theta_{N-2} , \\
x^4 &= \frac{2}{\sqrt{K}} \tan\frac{\sqrt{K}}{2} r \sin\theta_1 \sin\theta_2 \ldots \cos\theta_{N-3} , \quad \text{if } K \geq 0 \qquad \text{(a7.4)} \\
&\vdots \\
x^{N-1} &= \frac{2}{\sqrt{K}} \tan\frac{\sqrt{K}}{2} r \sin\theta_1 \cos\theta_2 , \\
x^N &= \frac{2}{\sqrt{K}} \tan\frac{\sqrt{K}}{2} r \cos\theta_1 ,
\end{aligned}
$$

and by

$$
\begin{aligned}
x^1 &= \frac{2}{\sqrt{-K}} \operatorname{th}\sqrt{-K} r \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{N-3} \sin\theta_{N-2} \sin\phi , \\
x^2 &= \frac{2}{\sqrt{-K}} \operatorname{th}\sqrt{-K} r \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{N-3} \sin\theta_{N-2} \sin\phi , \\
x^3 &= \frac{2}{\sqrt{-K}} \operatorname{th}\sqrt{-K} r \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{N-3} \cos\theta_{N-2} , \\
x^4 &= \frac{2}{\sqrt{-K}} \operatorname{th}\sqrt{-K} r \sin\theta_1 \sin\theta_2 \ldots \cos\theta_{N-3} , \quad \text{if } K \leq 0 \qquad \text{(a7.5)} \\
&\vdots \\
x^{N-1} &= \frac{2}{\sqrt{-K}} \operatorname{th}\sqrt{-K} r \sin\theta_1 \cos\theta_2 , \\
x^N &= \frac{2}{\sqrt{-K}} \operatorname{th}\sqrt{-K} r \cos\theta_1 .
\end{aligned}
$$

Remarks:

1. The polar coordinates $(r, \theta_1, \ldots, \theta_{N-2}, \phi)$ are defined in [40] by the coordinate transformations (a7.4) and (a7.5), then the metric forms (a7.2) and (a7.3) are deduced through substituting for $x^1, \ldots, x^N$ from (a7.4) and (a7.5) in (3.4.2).

2. The metric form (a7.2) and the transformation (a7.4) are formally identical to the metric form (a7.3) and to the transformation (a7.5) respectively if we allow K to take positive and negative values and make use of the identities

$$
\begin{aligned}
\sin z &= -i \operatorname{sh} iz \qquad & \operatorname{sh} z &= i \sin iz, \\
\tan z &= -i \operatorname{th} iz \qquad & \operatorname{th} z &= i \tan iz,
\end{aligned}
\qquad \text{(a7.6)}
$$

3. If we let $\sqrt{eK}\, r = \theta_0$, then the metric forms (a7.2) and (a7.3) may be written as follows:

$$ds^2 = \frac{1}{K}(d\theta_0^2 + \sin^2\theta_0\, d\theta_1^2 + \cdots + \sin^2\theta_0 \sin^2\theta_1 \cdots \sin^2\theta_{N-2}\, d\phi^2) \qquad (K>0)$$
$$ds^2 = \frac{1}{-K}(d\theta_0^2 + \operatorname{sh}^2\theta_0\, d\theta_1^2 + \cdots + \operatorname{sh}^2\theta_0 \sin^2\theta_1 \cdots \sin^2\theta_{N-2}\, d\phi^2) \qquad (K\ 0) \tag{a7.7}$$

where

$$0 \leqslant \theta_1, \cdots, \theta_{N-2} \leqslant \pi, \quad 0 \leqslant \phi < 2\pi,$$
$$0 \leqslant \theta_0 < \pi \ \text{ if } K>0 \text{ and } \theta_0 \geqslant 0 \text{ if } K<0. \tag{a7.8}$$

APPENDIX 8

Motions of CC^N

The OPG U^1 generated by L_1 is obtained through integrating the system of differential equations

$$\frac{dx^1}{\frac{K}{4}\left[(x^1)^2-\sum_{j=2}^{N}(x^j)^2\right]+1} = \frac{dx^2}{\frac{K}{2}x^1x^2} = \ldots = \frac{dx^N}{\frac{K}{2}x^1x^N} = dt . \tag{a8.1}$$

This above system gives

$$\frac{x^1\,dx^1}{\frac{K}{4}\left[(x^1)^2-\sum_{j=2}^{N}(x^j)^2\right]+1} = \frac{dx^2}{\frac{K}{2}x^2} = \ldots = \frac{dx^N}{\frac{K}{2}x^N}$$

$$\Longrightarrow \quad \frac{\frac{1}{2}x^\ell\,dx^\ell}{\frac{K}{4}x^\ell x^\ell+1} = \frac{dx^2}{Kx^2} \qquad = \frac{dx^N}{Kx^N}$$

$$\Longrightarrow \quad \begin{cases} x^\ell x^\ell + \frac{4}{K} = \alpha\, x^2 , \\ x^3 = \beta^3\, x^2 , \\ \vdots \\ x^N = \beta^N\, x^2 . \end{cases} \tag{a8.2}$$

Eliminating $x^3, \ldots, x^N$ between (a8.2) and substituting for x^1 in terms of x^2 in the differential equation $dt = dx^2/(\frac{K}{2}x^1x^2)$, we get

$$\frac{K}{2}\,dt = \frac{dx^2}{x^2\sqrt{-(1+\sum_{i=3}^{N}\beta_i^2)(x^2)^2+\alpha x^2-\frac{4}{K}}} . \tag{a8.3}$$

Setting $w=1/x^2$,

$$\frac{K}{2}\,t = -\int\frac{d\omega}{\sqrt{-\frac{4}{K}\omega^2+\alpha\omega-(1+\sum_{i=3}^{N}\beta_i^2)}} \tag{a8.4}$$

Case I $K<0$:

$$\frac{K}{2}\,t = -\frac{\sqrt{-K}}{2}\,\ell n\left|\frac{1}{\gamma}\left\{\frac{4}{\sqrt{-K}}\sqrt{-\frac{4}{K}\omega^2+\alpha\omega-(1+\sum_{i=3}^{N}\beta_i^2)}-\frac{8}{K}\omega+\alpha\right\}\right|$$

Substituting for w by $1/x^2$ and for α and β_i from (a8.2),

$$\gamma e^{\sqrt{-K}t} = \frac{4\sqrt{-K}x^1+4-Kx^\ell x^\ell}{-Kx^2} \tag{a8.5}$$

Solving (a8.2) and (a8.5) for $x^1, \ldots, x^N$ in terms of $\alpha, \beta_2, \ldots, \beta_N, \gamma$ we find

$$\bar{x}^1 = -\frac{\sqrt{-K}}{4}\,\frac{\gamma e^{\sqrt{-K}t}(\alpha-\gamma e^{\sqrt{-K}t})}{-\frac{K}{16}(\alpha-\gamma e^{\sqrt{-K}t})^2+1+\sum_{i=3}^{N}\beta_i^2}-\frac{2}{\sqrt{-K}},$$

$$\bar{x}^2 = \frac{\gamma e^{\sqrt{-K}t}}{-\frac{K}{4}(\alpha-\gamma e^{\sqrt{-K}t})^2+1+\sum_{i=3}^{N}\beta_i^2}, \tag{a8.6}$$

$$\bar{x}^i = \beta_i\,\bar{x}^2 \quad (i=3,\ldots,N),$$

where

$$\alpha = (x^\ell x^\ell+\frac{4}{K})/x^2,\quad \beta_i = x^i/x^2,\quad \gamma = (4\sqrt{-K}x^1+4-Kx^\ell x^\ell)/(-Kx^2) \tag{a8.7}$$

Case II $K>0$:

$$\frac{K}{2}\,t = \gamma' + \frac{\sqrt{K}}{2}\arcsin\frac{-\frac{8}{K}\omega+\alpha}{\sqrt{\alpha^2-\frac{16}{K}(1+\sum_{i=3}^{N}\beta_i^2)}} \tag{a8.8}$$

Substituting for w by $1/x^2$ and for $\alpha, \beta_3, \ldots, \beta_N$ from (a8,2) in (a8.8)',

$$\sin(\sqrt{K}t-\gamma) = \frac{K x^\ell x^\ell - 4}{\sqrt{(K x^\ell x^\ell - 4)^2 + 16K(x^1)^2}} \quad . \tag{a8.9}$$

Solvimg for $x^1, \ldots, x^N$ from (a8.2) and (a8.9) we find

$$\bar{x}^1 = \frac{2}{\sqrt{K}} \frac{\sqrt{\alpha^2K^2 - 16K(1+\sum_{i=3}^{N}\beta_i^2)}\ \cos(\sqrt{K}t-\gamma)}{\alpha K - \sqrt{\alpha^2K^2 - 16K(1+\sum_{i=3}^{N}\beta_i^2)}\ \sin(\sqrt{K}t-\gamma)} \quad ,$$

$$\bar{x}^2 = \frac{8}{\alpha K - \sqrt{\alpha^2K^2 - 16K(1+\sum_{i=3}^{N}\beta_i^2)}\ \sin(\sqrt{K}t-\gamma)} \quad ,$$

$$\bar{x}^i = \beta_i\,\bar{x}^2 \quad ,$$

where

$$\alpha = (x^\ell x^\ell + \frac{4}{K})/x^2 \quad , \quad \beta_i = x^i/x^2 \quad ,$$

$$\sin\gamma = \frac{4 - K x^\ell x^\ell}{\sqrt{(K x^\ell x^\ell - 4)^2 + 16K(x^1)^2}} \quad , \quad \cos\gamma = \frac{4\sqrt{K}\ x^1}{\sqrt{(K x^\ell x^\ell - 4)^2 + 16K(x^1)^2}} \quad .$$

The rest of OPG's $U^2, \ldots, U^N$ are obtained through utilizing the symmetry between the system (a8.1) and the system corresponding to U^i (i=2, ... ,N).

The OPG's U^{ij} (i=j;i,j=1, ... ,N) are easily derived.

APPENDIX 9

Evaluation of the Eigenfunctions of the Momenta in CC^N

The eigenfunctions of a momentum operator $\hat{P} = -ih\ \xi^i \partial/\partial x^i$ on the solutions of the quasi-linear partial differential equation

$$-i\hbar\ \xi^i \frac{\partial \psi}{\partial x^i} = \lambda \psi . \tag{a9.1}$$

Following the standard method for solving such an equation, we set

$$\frac{dx^1}{-i\hbar \xi^1} = \ldots = \frac{dx^N}{-i\hbar \xi^N} = \frac{d\psi}{\lambda \psi} = \frac{dt}{-i\hbar} . \tag{a9.2}$$

If $f_i = c_i$ ($i = 1,\ldots,N$; c_i are arbitrary constants) are N-independent integrals for (a9.2), then the general solution of (a9.1) is given by

$$F_0(f_1, \ldots, f_N) = 0 ,$$

where F_0 is an arbitrary function in its arguments. From $\frac{d\psi}{\lambda\psi} = \frac{dt}{-ih}$, we have $f_n \equiv \psi\ e^{-i\lambda t/h} = c_n$. Thus the general solution of (a9.1) may be put in the form

$$\psi_\lambda = e^{i\lambda t/\hbar} F(f_1, \ldots, f_N) , \tag{a9.3}$$

where F is an arbitrary function and λ is the eigenvalue corresponding to the eigenfunction ψ_λ.

The integrals $f_1 = c_1, \ldots, f_{N-1} = c_{N-1}$ are known already when the system

$$\frac{dx^1}{\xi^1} = \ldots = \frac{dx^N}{\xi^N} = dt \tag{a9.4}$$

was solved in the course of finding the motions of CC^N. The integrals $f_i = c_i$ $(i = 1,\ldots,N-1)$ are the equations of the integral curves σ of L. On σ we have

$$\begin{aligned}\psi_\lambda(\sigma(t)) &= F(c_1,\ldots,c_{N-1})\, e^{i\lambda t/\hbar} \\ &= \psi_\lambda(\sigma(0))\, e^{i\lambda t/\hbar},\end{aligned} \tag{a9.5}$$

which is just what proposition (7.2) states.

Also when we solved the system (a9.4) we found an N-th integral of the form $t - c' = g_N(\underline{x})$. Substituting for t in (a9.3),

$$\psi_\lambda = e^{i\lambda g_N(\underline{x})/\hbar}\, F(f_1,\ldots,f_{N-1})\,. \tag{a9.6}$$

(The constant $e^{-i\lambda c'}$ has been absorbed in F).

In what follows we write down explicitly the eigenfunctions of the momentum operators in CC^N.

1. Eigenfunctions of $\hat{M}_{kj}$

From (3.8.10), we find the following N-independent integrals:

$$x^r = c^r \quad (r \neq k,\ r \neq j), \tag{a9.7a}$$

$$(x^k)^2 + (x^j)^2 = a^2 \quad , \tag{a9.7b}$$

$$e^{i(t-t_0)} = \frac{x^k + i x^j}{\sqrt{(x^k)^2 + (x^j)^2}} \tag{a9.7c}$$

Thus

$$\psi_\lambda = \left(\frac{x^k + i x^j}{\sqrt{(x^k)^2 + (x^j)^2}}\right)^{\lambda/\hbar} F(\underline{x}^r_{kj}, (x^k)^2 + (x^j)^2), \tag{a9.8}$$

where

$$(\underline{x}^r_{kj}) = (x^1,\ldots,x^{k-1},x^{k+1},\ldots,x^{j-1},x^{j+1},\ldots,x^N)\,.$$

By (a9.8) and (a9.7c),

$$\begin{aligned}&\psi(\underline{c}^r_{kj}, a^2, 0) = \psi(\underline{c}^r_{kj}, a^2, 2m\pi) \qquad (m \in \mathbb{N}) \\ \Rightarrow\ &F(\underline{c}^r_{kj}, a^2) = F(\underline{c}^r_{kj}, a^2)\, e^{i\lambda 2m\pi/\hbar} \\ \Rightarrow\ &\lambda = n\hbar \quad (n \in \mathbb{N})\,.\end{aligned}$$

Therefore the eigenfunctions are

$$\psi_n = \left(\frac{x^k + i x^j}{\sqrt{(x^k)^2+(x^j)^2}}\right)^n F(\underline{x}^r_{kj}, (x^k)^2+(x^j)^2) . \tag{a9.9}$$

2. Eigenfunctions of $\hat{N}_1$

(i) Case K < 0

From (a8.2) and (a8.5),

$$(Kx^\ell x^\ell + 4)/(Kx^2) = c^1 , \tag{a9.10a}$$

$$x^r/x^2 = c^{r-1} \quad (r=3,\ldots,N) , \tag{a9.10b}$$

$$e^{\sqrt{-K}(t-t_0)} = (4\sqrt{-K}\,x^1 + 4 - Kx^\ell x^\ell)/(-Kx^2) . \tag{a9.10c}$$

Thus

$$\psi_\lambda = \left(\frac{4\sqrt{-K}\,x^1+4-Kx^\ell x^\ell}{-Kx^2}\right)^{i\lambda/\hbar\sqrt{-K}} F\left(\frac{Kx^\ell x^\ell+4}{Kx^2}, \frac{\underline{x}^r}{x^2}\right), \tag{a9.11}$$

where $\lambda \in \mathbb{R}$ and

$$(\underline{x}^r/x^2) = (x^3/x^2, \ldots, x^N/x^2) .$$

(ii) Case K > 0

From (a8.2) and (a8.9),

$$(Kx^\ell x^\ell + 4)/(Kx^2) = c^1 , \tag{a9.12a}$$

$$x^r/x^2 = c^{r-1} \quad (r=3,\ldots,N) , \tag{a9.12b}$$

$$e^{i\sqrt{K}(t-t_0)} = \frac{4\sqrt{K}\,x^1 + i(4-Kx^\ell x^\ell)}{\sqrt{16K(x^1)^2+(4-Kx^\ell x^\ell)^2}} . \tag{a9.12c}$$

Thus

$$\psi_\lambda = \left(\frac{4\sqrt{K}\,x^1+i(4-Kx^\ell x^\ell}{\sqrt{16(x^1)^2+(Kx^\ell x^\ell-4)^2}}\right)^{\lambda/\sqrt{K}\hbar} F\left(\frac{Kx^\ell x^\ell+4}{Kx^2}, \frac{\underline{x}^r}{x^2}\right). \tag{a9.13}$$

By (a9.12c) and (a9.13),

$$\psi(c^1,\ldots,c^{N-1},0) = \psi(c^1,\ldots,c^{N-1},2\pi/\sqrt{K})$$

$$\Longrightarrow \lambda = n\sqrt{K}\,\hbar \qquad (n\in\mathbb{N}) .$$

(iii) K = 0

It is easily found in this case that

$$\psi_\lambda = F(\underline{x}_1^r)\, e^{i\lambda x^1/\hbar} \qquad (\lambda \in \mathbb{R}), \tag{a9.14}$$

where

$$(\underline{x}_1^r) = (x^2, \ldots, x^N) .$$

CHAPTER IV

Spaces of Constant Curvature as Hypersurfaces of a Flat Space.

§1. Introduction.

A flat space is a space for which all the components of the Riemannian tensor R_{ijks} vanish. Equivalently, it is a space whose Riemannian curvature K vanishes [34]. There exists in an (N+1)-dimensional flat space F^{N+1} a coordinate system $(X^1, \ldots, X^{N+1})$ such that the metric assumes the form

$$ds^2 = \sum_{\nu=1}^{N+1} \varepsilon_\nu (dX^\nu)^2, \tag{4.1.1}$$

where $\varepsilon_\nu (\nu=1, \ldots, N+1)$ are +1 or -1 [34]. We call such coordinates Cartesian.

Let $(x^1, \ldots, x^N)$ be a coordinate system in a hypersurface in a F^{N+1}. Let this surface be given parametrically by

$$X^i = f^i(x^1, \ldots, x^N) \quad (i=1, \ldots, N+1) \tag{4.1.2}$$

It is easily seen that the components g_{ij} of the metric of this hypersurface are related to the components of the metric of F^{N+1} by the relations [34]

$$g_{ij} = \sum_{\nu=1}^{N+1} \varepsilon_\nu \frac{\partial X^\nu}{\partial x^i} \frac{\partial X^\nu}{\partial x^j} \quad . \tag{4.1.3}$$

The hypersurfaces defined by

$$\sum_{\nu=1}^{N+1} \varepsilon_\nu (X^\nu)^2 = eR^2, \tag{4.1.4}$$

where e is either +1 or -1, are called the fundamental hyperquadratics of F^{N+1}. If $\varepsilon_\nu=1$ for all ν, then the metric (4.1.1) takes the form

$$ds^2 = dX^\nu \, dX^\nu . \tag{4.1.5}$$

F^{N+1} in this case is the Euclidean space E^{N+1}. Such a space possesses only one family of fundamental hyperquadratics, namely the hyperspheres

$$X^\nu X^\nu = R^2 . \tag{4.1.6}$$

If $\varepsilon_\nu=1$ for $\nu=1, \ldots ,N$ and $\varepsilon_{N+1}=-1$, then the metric (4.1.1) is the Minkowskian metric

$$ds^2 = dX^i \, dX^i - (dX^{N+1})^2 \quad (i=1, \ldots ,N) \tag{4.1.7}$$

and F^{N+1} in this case is the Minkowski space M^{N+1}. The Minkowski space M^{N+1} possesses two families of fundamental hyperquadratics

$$X^i X^i - (X^{N+1})^2 = R^2 , \tag{4.1.8}$$

and

$$X^i X^i - (X^{N+1})^2 = -R^2 . \tag{4.1.9}$$

The following two propositions [34] identify a CC^N as a fundamental hyperquadratic of F^{N+1}.

Proposition (1.1). The fundamental hyperquadratics of F^{N+1} are CC^N with curvature $K = e/R^2$.

Proposition (1.2). The fundamental hyperquadratics are the only hypersurfaces of non-vanishing constant Riemannian curvature of F^{N+1}.

The above considerations show that while a CC_+^N can be embedded as a hypersurface in E^{N+1}, a CC_-^N cannot be embedded in E^{N+1}. However, a CC_-^N can be embedded in M^{N+1} as what we may call a pseudo-hypersphere,

§2. CC^N Embedded in an (N+1)-Dimensional Flat Space.

Let V_K be a CC^N with curvature $K \neq 0$, and $(x^1, \dots, x^N)$ be a coordinate system in V_K in terms of which the metric assumes the Riemannian form (3.4.2). If $K>0$, then the equations

$$X^i = x^i/(1+\frac{K}{4}x^s x^s) \quad (i=1, \dots, N),$$
$$X^{N+1} = (1/\sqrt{K})(1-\frac{K}{4}x^s x^s)/(1+\frac{K}{4}x^s x^s) \quad , \tag{4.2.1}$$

provide a parametric representation of V_K as a hypersurface in E^{N+1}. If $K<0$, then V_K can be embedded in M^{N+1} as the pseudo-hypersphere

$$X^i = x^i/(1+\frac{K}{4}x^s x^s) \quad (i=1, \dots, N)$$
$$X^{N+1} = (1/\sqrt{-K})(1-\frac{K}{4}x^s x^s)/(1+\frac{K}{4}x^s x^s). \tag{4.2.2}$$

From (4.2.1) and (4.2.2),

$$\partial/\partial x^r = (1+\frac{K}{4}x^s x^s)^{-2}\Big\{-\frac{K}{2}\sum_{\substack{i=1\\ i\neq r}}^{N} x^r x^i\,\partial/\partial X^i + [1+\frac{K}{4}(\sum_{\substack{i=1\\ i\neq r}}^{N}(x^i)^2-(x^r)^2)]\,\partial/\partial X^r$$
$$-e\sqrt{eK}\,x^r\,\partial/\partial X^{N+1}\Big\} \quad (r=1, \dots, N), \tag{4.2.3}$$

where $eK=|K|$ and no summation over r is implied. By (4.2.3), (4.2.1) and (4.2.2), the infinitesimal motions L_{ij} and L_i of V_K given by (3.8.8) can be written in the following forms:

$$L_{ij} = \mathcal{L}_{ij}(V_K) \quad ; L_i = \sqrt{K}\,\mathcal{L}_{N+1\ i}(V_K) \quad \text{if} \quad K>0$$
$$= \sqrt{-K}\,B_i(V_K) \qquad \text{if} \quad K<0, \tag{4.2.4}$$

where

$$\mathcal{L}_{rs} = X^r\partial/\partial X^s - X^s\partial/\partial X^r \ , \ B_i = X^{N+1}\,\partial/\partial X^i + X^i\,\partial/\partial X^{N+1} \ . \tag{4.2.5}$$

and the argument V_K denotes the restriction of $\mathcal{L}_{rs}$ and B_i to V_K.

Remarks

1. In terms of $(X^1, \dots, X^{N+1})$, the Hamiltonian $\hat{H}$ may be written as

follows

$$\hat{H} = (K/2m) \sum_{\substack{r,s=1 \\ r<s}}^{N+1} \hat{\mathcal{L}}_{rs}^2 (V_K) \qquad \text{if} \quad K>0,$$
$$\hat{H} = (-K/2m)\left\{ \sum_{i=1}^{N} \hat{B}_i^2 (V_K) - \sum_{\substack{i,j=1 \\ i<j}}^{N} \hat{\mathcal{L}}_{ij}^2 (V_K) \right\} \quad \text{if } K<0 . \tag{4.2.6}$$

Here $\hat{\mathcal{L}}_{nm} = -i\hbar \mathcal{L}_{nm}$ and $\hat{B}_i = -i\hbar\, B_i$.

2. The OPG's U^{ij} generated by L_{ij} $(i,j=1, \ldots ,N)$ may be written in terms of $(X^1, \ldots ,X^{N+1})$ as

$$\begin{aligned} \overline{X}^i &= [(X^i)^2+(X^j)^2]^{1/2} \cos(t-t_\circ) , \\ \overline{X}^j &= [(X^i)^2+(X^j)^2]^{1/2} \sin(t-t_\circ) , \\ \overline{X}^r &= X^r \quad (r\neq i; r\neq j; r=1, \ldots ,N+1), \end{aligned} \tag{4.2.7}$$

where

$$\cos t_\circ = X^i/[(X^i)^2+(X^j)^2]^{1/2} , \quad \sin t_\circ = -X^j/[(X^i)^2+(X^j)^2]^{1/2} \tag{4.2.8}$$

The orbits σ^{ij} of U^{ij} are given by

$$\begin{aligned} &\overline{X}^s\overline{X}^s + e(\overline{X}^{N+1})^2 = 1/K \quad (s=1, \ldots ,N) , \\ &\overline{X}^r = X^r \quad (r\neq i; r\neq j; r=1, \ldots ,N+1), \end{aligned} \tag{4.2.9}$$

where $eK=|K|$. The OPG's U^i generated by L_i $(i=1, \ldots ,N)$ have the forms

Case $K>0$

$$\begin{aligned} \overline{X}^{N+1} &= [(X^{N+1})^2+(X^i)^2]^{1/2} \cos(\sqrt{K}\, t-t_\circ), \\ \overline{X}^i &= [(X^{N+1})^2+(X^i)^2]^{1/2} \sin(\sqrt{K}\, t-t_\circ), \\ \overline{X}^r &= X^r \quad (r\neq i; r=1, \ldots ,N) , \end{aligned} \tag{4.2.10}$$

where

$$\cos t_\circ = X^{N+1}/[(X^{N+1})^2+(X^i)^2]^{1/2}, \sin t_\circ = -X^i/[(X^{N+1})^2+(X^i)^2]^{1/2}; \tag{4.2.11}$$

Case $K<0$

$$\begin{aligned} \bar{X}^{N+1} &= [(X^{N+1})^2-(X^i)^2]^{1/2}\ \mathrm{ch}(\sqrt{-K}\ t-t_\circ), \\ \bar{X}^i &= [(X^{N+1})^2-(X^i)^2]^{1/2}\ \mathrm{sh}(\sqrt{-K}\ t-t_\circ), \\ \bar{X}^r &= X^r \quad (r\neq i;\ r=1,\ \dots\ ,N) \quad , \end{aligned} \tag{4.2.12}$$

where

$$\mathrm{sh}\ t_\circ = -X^i/[(X^{N+1})^2-(X^i)^2]^{1/2}. \tag{4.2.13}$$

The orbits of U^i $(i=1,\ \dots\ ,N)$ are given by

$$\begin{aligned} \bar{X}^s\bar{X}^s+e(\bar{X}^{N+1})^2 &= 1/K, \\ \bar{X}^r &= X^r \quad (r\neq i, r=1,\ \dots\ ,N), \end{aligned} \tag{4.2.14}$$

where $eK = |K|$.

3. In agreement with §(8.3) of ch. III, eqs. (4.2.7), (4.2.10) and (4.2.11) show that

(i) the orbits of U^{ij} are periodic with a period $T=2\pi$.

(ii) the orbits of U^i are non-periodic if $K<0$, but are periodic if $K>0$, with a period $T=2\pi/\sqrt{K}$.

4. Ignoring a multiplicative factor, the infinitesimal motions of V_K $(K>0)$ are the restrictions to V_K of the infinitesimal rotations $\mathcal{L}_{ij}$ $(i\neq j;i,j=1,\ \dots\ ,N+1)$ of E^{N+1} in the (X^i,X^j)-planes. The situation is different if $K<0$. While L_{ij} are the restrictions to V_K of the infinitesimal rotations $\mathcal{L}_{ij}$ $(i=j;\ i,j=1,\ \dots\ ,N)$ of M^{N+1} in the (X^i,X^j)-planes, the infinitesimal motions L_i $(i=1,\ \dots\ ,N)$ are of different nature; they are the restrictions of the infinitesimal homogeneous Lorentz transformations of M^{N+1} in the (X^{N+1},X^i)-planes.

5. Consider an N-dimensional manifold $V_K=\{(y^1,\ \dots\ ,y^N)\in\mathbb{R}^N | y^N>0\}$. Define a metric form in V_K by $ds^2= R^2\ dy^i\ dy^i/(y^N)^2$. The manifold V_K with this metric is a CC^N_- with curvature $K = -1/R^2$. V_K can be

embedded in M^{N+1} by the equations

$$X^i = (1/\sqrt{-K})\ y^i/y^N \quad ,$$

$$X^N = (1/\sqrt{-K})(y^s y^s - 1)/(2y^N),$$

$$X^{N+1} = (1/\sqrt{-K})(y^s y^s + 1)/(2y^N),$$

where s is summed from 1 to N.

§3. Special Cases.

We shall consider the problem of embedding of CC^2 when endowed with parallel geodesic or polar geodesic coordinates. Also, we shall discuss the embedding of CC^3 when endowed with a geodesic polar coordinate system.

§(3.1) The Two-Dimensional Case.

§(3.1.1) CC^2_- Embedded in M^3.

Let V_K be a CC^2_- whose curvature is K. V_K may be envisaged as a branch of the pseudo-sphere $(X^1)^2+(X^2)^2-(X^3)^2= +1/K$ in M^3. The geodesics of such a space are its lines of intersection with planes through the origin $(0,0,0)$ [27]. The upper branch of this pseudo-sphere may be represented by the equations

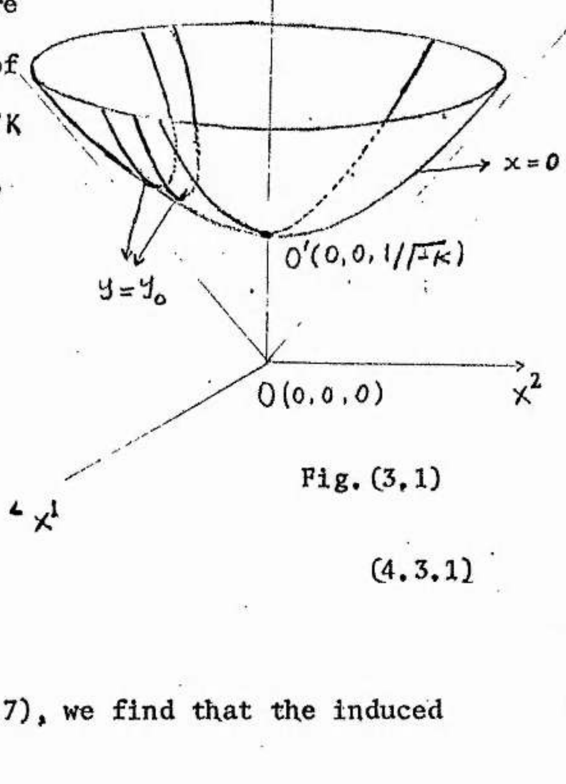


Fig. (3.1)

$$
\begin{aligned}
X^1 &= (1/\sqrt{-K})\ \mathrm{sh}\sqrt{-K}\, x , \\
X^2 &= (1/\sqrt{-K})\ \mathrm{ch}\sqrt{-K}\, x\ \mathrm{sh}\sqrt{-K}\, y, \\
X^3 &= (1/\sqrt{-K})\ \mathrm{ch}\sqrt{-K}\, x\ \mathrm{ch}\sqrt{-K}\, y,
\end{aligned}
\tag{4.3.1}
$$

where $-\infty<x,y<\infty$. From (4.3.1) and (4.1.7), we find that the induced metric from M^3 in V_K is

$$ds^2 = dx^2 + \mathrm{ch}^2\sqrt{-K}\, x\ dy^2. \tag{4.3.2}$$

Thus, the chart (x,y) forms a geodesic coordinate system in V_K. The centre of this chart is $(0,0,1/\sqrt{-K})$, and the geodesic

$$x = 0 \Leftrightarrow \begin{cases} X^1 = 0 \\ (X^1)^2+(X^2)^2-(X^3)^2 = 1/K, \end{cases}$$

is orthogonal to the one-parameter family of geodesics

$$y = y_\circ \Longleftrightarrow \begin{cases} X^2 = X^3 \ \mathrm{th}\sqrt{-K}\, y_\circ \\ (X^1)^2 + (X^2)^2 - (X^3)^2 = 1/K, \end{cases}$$

which are the lines of intersections of planes through OX^1 with V_K (Fig. (3.1)).

From (4.3.1) we have

$$\partial/\partial x = \mathrm{ch}\sqrt{-K}\,x\ \partial/\partial X^1 + \mathrm{sh}\sqrt{-K}\,x\ \mathrm{sh}\sqrt{-K}\,y\ \partial/\partial X^2 + \mathrm{sh}\sqrt{-K}\,x\ \mathrm{ch}\sqrt{-K}\,y\ \partial/\partial X^3,$$
$$\partial/\partial y = \mathrm{ch}\sqrt{-K}\,x\ \mathrm{ch}\sqrt{-K}\,y\ \partial/\partial X^2 + \mathrm{ch}\sqrt{-K}\,x\ \mathrm{sh}\sqrt{-K}\,y\ \partial/\partial X^3 \quad . \qquad (4.3.3)$$

By (4.3.3) and (4.3.1), the momenta $\hat{P}_\mu$ (μ=1,2,3) in V_K given by (3.5.9) and (3.5.12) take the forms

$$\hat{P}_1 = \sqrt{-K}\ \hat{B}_1(V_K), \quad \hat{P}_2 = \sqrt{-K}\ \hat{B}_2(V_K), \quad \hat{P}_3 = \hat{\mathcal{L}}_{12}(V_K). \qquad (4.3.4)$$

The Hamiltonian in V_K, given by (3.5.14), may be written in the form

$$\hat{H} = (-K/2m)\,(\hat{B}_1^2(V_K) + \hat{B}_2^2(V_K) - \hat{\mathcal{L}}_{12}^2(V_K))$$

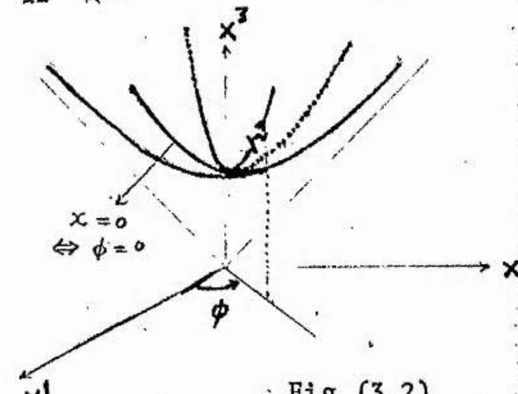


Fig. (3.2)

If (r,ϕ) is a geodesic polar coordinate system in V_K, then V_K may be represented by

$$X^1 = (1/\sqrt{-K})\ \mathrm{sh}\sqrt{-K}\ r\cos\phi\ ,$$
$$X^2 = (1/\sqrt{-K})\ \mathrm{sh}\sqrt{-K}\ r\sin\phi\ , \qquad (4.3.5)$$
$$X^3 = (1/\sqrt{-K})\ \mathrm{ch}\sqrt{-K}\ r \quad .$$

The momenta $\hat{P}_\mu$ (μ=1,2,3) in V_K given by (3.5.47) can be written in terms of (X^1, X^2, X^3) in the form (4.3.4).

From (4.3.1) and (4.3.5) we can deduce the coordinate transformation between the (r,ϕ) system and the (x,y) system. The centre of the system (r,ϕ) is $(0,0,1/\sqrt{-K})$, and ϕ is measured in an anti-clockwise direction from OX^1. If we replace ϕ by $(\frac{\pi}{2} - \phi)$ in (4.3.5), then $\phi = 0$

coincide with $x = 0$ (Fig. (3.3)). The coordinate transformation between the two systems (in this new position) is

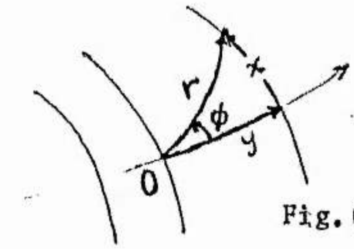


Fig. (3,3)

$$x = (\varepsilon/\sqrt{-K})\ \mathrm{Argch}(\mathrm{ch}^2\sqrt{-K}\ r - \mathrm{sh}^2\sqrt{-K}\ r\cos^2\phi)^{1/2},$$
$$y = (1/\sqrt{-K})\ \mathrm{Argth}(\mathrm{th}\sqrt{-K}\ r\cos\phi),$$

where $\varepsilon=1$ if $\phi\in[0,\pi)$ and $\varepsilon=-1$ if $\phi\notin[0,\pi)$. Since the coordinate system (x,y) is global, the above coordinate transformation is 1-1 in the domain of the chart (r,ϕ). The inverse transformation is given by

$$r = (1/\sqrt{-K})\ \mathrm{Argsh}(\mathrm{sh}^2\sqrt{-K}\ x + \mathrm{ch}^2\sqrt{-K}\ x\ \mathrm{sh}^2\sqrt{-K}\ y)^{1/2},$$
$$\tan\phi = \mathrm{th}\sqrt{-K}\ x\ \mathrm{sh}\sqrt{-K}\ y,$$

where $\phi\in[0,\frac{\pi}{2}]$ if $(x\geqslant 0, y\geqslant 0)$, $\phi\in[\frac{\pi}{2},\pi]$ if $(x\geqslant 0, y\leqslant 0)$, $\phi\in[\pi,\frac{3\pi}{2}]$ if $(x\leqslant 0, y\leqslant 0)$ and $\phi\in[\frac{3\pi}{2},2\pi)$ if $(x\geqslant 0, y<0)$.

The hyperbolic plane V_{-1} as the unit pseudo-sphere in M^3 can be represented by the following equations:

$$X^1 = y^1/y^2,$$
$$X^2 = (1/2y^2)((y^1)^2+(y^2)^2-1),$$
$$X^3 = (1/2y^2)((y^1)^2+(y^2)^2+1),$$

Here the coordinates (x,y) which were used in §7 in the last chapter are denoted by (y^1,y^2). It can be verified that in terms of (X^1,X^2,X^3), the quantum momentum operators given by (3,7,3) and (3,7,4) take the forms

$$\hat{P}_1 = \hat{B}_2(V_{-1}),\quad \hat{P}_2 = -\hat{B}_1(V_{-1}),\quad \hat{P}_3 = \hat{\mathcal{L}}_{12}(V_{-1}).$$

§(3.1.2) CC_+^2 Embedded in E^3.

Consider an one-parameter family of spheres in E^3 with a common centre $(0,0,0)$ and radii ρ. Let (x,y) be a

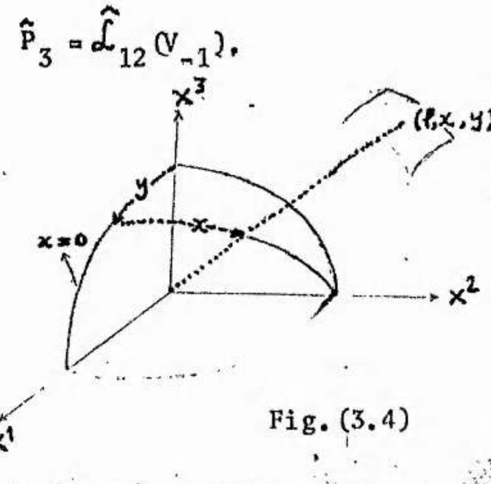


Fig. (3.4)

geodesic coordinate system in some member of this family say in the sphere whose curvature is K. In terms of the normal coordinate system (ρ,x,y) in E^3 [35], the metric form of E^3 is

$$ds^2 = d\rho^2 + \rho^2 K(dx^2 + \cos^2\sqrt{K}\,x\,dy^2). \tag{4.3.7}$$

A possible coordinate transformation between (ρ,x,y) and (X^1,X^2,X^3) is

$$\begin{aligned} X^1 &= \rho\cos\sqrt{K}\,x\,\sin\sqrt{K}\,y, \\ X^2 &= \rho\sin\sqrt{K}\,x, \\ X^3 &= \rho\cos\sqrt{K}\,x\,\cos\sqrt{K}\,y, \end{aligned} \tag{4.3.8}$$

Let V_K be a CC_+^2 with curvature K. Setting $\rho = 1/\sqrt{K}$ in (4.3.7) we recover the metric form of V_K in the geodesic coordinate system (x,y). Parametrically, V_K may be represented by (4.3.8) through setting $\rho = 1/\sqrt{K}$.

V_K can also be represented by

$$\begin{aligned} X^1 &= (1/\sqrt{K})\,\sin\sqrt{K}\,r\,\cos\phi, \\ X^2 &= (1/\sqrt{K})\,\sin\sqrt{K}\,r\,\sin\phi, \\ X^3 &= (1/\sqrt{K})\,\cos\sqrt{K}\,r, \end{aligned}$$

where (r,ϕ) is a geodesic polar coordinate system in V_K.

It can be shown that the set of momenta in V_K expressed in terms of (x,y) or (r,ϕ) [§(5.2), §(5.4.4)] can be written in terms of (X^1,X^2,X^3) as

$$\hat{P}_1 = \sqrt{K}\,\hat{\mathcal{L}}_{23}(V_K), \quad \hat{P}_2 = \sqrt{K}\,\hat{\mathcal{L}}_{31}(V_K), \quad \hat{P}_3 = \hat{\mathcal{L}}_{12}(V_K).$$

The Hamiltonian in V_K in terms of (X^1,X^2,X^3) is

$$\hat{H} = \frac{K}{2m}\left\{\hat{\mathcal{L}}^2_{23}(V_K) + \hat{\mathcal{L}}^2_{31}(V_K) + \hat{\mathcal{L}}^2_{12}(V_K)\right\} = \frac{K}{2m}\,\mathcal{L}^2(V_K),$$

where $\mathcal{L}^2$ is the square of the total angular momentum operator in V_K.

§(3.2) The Three-Dimensional Case.

Let (r,ϕ,θ) be a geodesic polar coordinate system in V_K, where V_K is a CC^3. If $K>0$, then it is clear that the equations

$$\begin{aligned} X^1 &= (1/\sqrt{K})\,\sin\sqrt{K}\,r\,\sin\theta\,\cos\phi\,, \\ X^2 &= (1/\sqrt{K})\,\sin\sqrt{K}\,r\,\sin\theta\,\sin\phi\,, \\ X^3 &= (1/\sqrt{K})\,\sin\sqrt{K}\,r\,\cos\theta\,, \\ X^4 &= (1/\sqrt{K})\,\cos\sqrt{K}\,r\,, \end{aligned} \tag{4.3.10}$$

form a parametric representation of V_K as a 3-sphere in E^4. If $K<0$, then V_K may be represented as the upper branch of a 3-pseudo-sphere in M^3 by

$$\begin{aligned} X^1 &= (1/\sqrt{-K})\,\mathrm{sh}\sqrt{-K}\,r\,\sin\theta\,\cos\phi\,, \\ X^2 &= (1/\sqrt{-K})\,\mathrm{sh}\sqrt{-K}\,r\,\sin\theta\,\sin\phi\,, \\ X^3 &= (1/\sqrt{-K})\,\mathrm{sh}\sqrt{-K}\,r\,\cos\theta\,, \\ X^4 &= (1/\sqrt{-K})\,\mathrm{ch}\sqrt{-K}\,r\,. \end{aligned} \tag{4.3.11}$$

Routine calculations show that the momenta $\hat{N}_i$ and $\hat{M}_i$ in CC^3 given by (3.6.35) can be written in terms of the coordinate system (X^1,X^2,X^3,X^4) in F^4 ($F^4 \equiv E^4$ if $K>0$, $F^4 \equiv M^4$ if $K<0$) as

$$\begin{aligned} &\hat{M}_1 = \hat{\mathcal{L}}_{23}\,,\ \hat{M}_2 = \hat{\mathcal{L}}_{31}\,,\ \hat{M}_3 = \hat{\mathcal{L}}_{12}\,, \\ &\hat{N}_1 = \sqrt{K}\,\hat{\mathcal{L}}_{14}\,,\ \hat{N}_2 = \sqrt{K}\,\hat{\mathcal{L}}_{24}\,,\ \hat{N}_3 = \sqrt{K}\,\hat{\mathcal{L}}_{34} \quad K>0, \\ &\hat{N}_1 = \sqrt{-K}\,\hat{B}_1\,,\ \hat{N}_2 = \sqrt{-K}\,\hat{B}_2\,,\ \hat{N}_3 = \sqrt{-K}\,\hat{B}_3 \quad K<0, \end{aligned} \tag{4.3.12}$$

§4. Dirac Theory on Systems under Constraints.

It has been demonstrated that spaces of constant curvature CC^N may be regarded as hypersurfaces in a higher dimensional flat space F^{N+1}. The question naturally arises as to whether the problem of quantization in CC^N may be described as the quantization in F^{N+1} under geometric constraints. The best known quantization scheme dealing with constraints is perhaps the theory of Dirac, Consider a classical system described by the Lagrangian $\mathcal{L}(X^i, \dot{X}^i)$. The Hamiltonian is

$$H(\underline{X},\underline{p}) = p_i\dot{X}^i - \mathcal{L}, \tag{4.4.1}$$

where $p_i = \partial\mathcal{L}/\partial\dot{X}^i$. Now if p_i so defined turn out not to be independent, i.e. p_i satisfy some equations

$$\varphi_r(\underline{X},\underline{p}) = 0 \quad (r=1, \ldots, m), \tag{4.4.2}$$

then the system is said to be under constraints. The equations of motion are obtained through the variation of the Hamiltonian subject to the constraints $\varphi_r = 0$; they are

$$\begin{aligned} \dot{X}^i &= \frac{\partial H}{\partial p_i} + u_r\frac{\partial \varphi_r}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial X^i} - u_r\frac{\partial \varphi_r}{\partial X^i}, \end{aligned} \tag{4.4.3}$$

where u_r are unknown coefficients. The quantity

$$H_T = H + u_r\,\varphi_r \tag{4.4.4}$$

is called the total Hamiltonian, For the system to be consistent dynamically, i.e. φ_r remain zero all the time we must have

$$d\varphi_r/dt = \{\varphi_r, H_T\} = \{\varphi_r, H\} + u_{r'}\{\varphi_r, \ _{r'}\} = 0. \tag{4.4.5}$$

These above consistency requirements may impose some conditions on u_r, or may lead to further conditions like (4.4.2). We call these conditions, $\phi_k(\underline{X},\underline{p})=0$, secondary constraints. Dirac devides the constraints into first and second class. A constraint (primary or secondary) is first class if its Poisson brackets with all other constraints vanish. A constraint is second class if it is not first class. Let $\eta_m(\underline{X},\underline{p})=0$, $\chi_s(\underline{X},\underline{p})=0$ denote first and second class constraints respectively. Observe that a constraint expression, say $\varphi_r(\underline{X},\underline{p})=0$, cannot be put to zero before a Poisson bracket is calculated, that is, a constraint cannot be regarded simply as an identity which may be put to zero whenever we like it. But if we replace the usual Poisson brackets by a new kind of brackets (to be called Dirac bracket)[47] then two major results emerge:

(1) Equations of motion remain formally the same as before, i,e,

$$df(\underline{X},\underline{p})/dt = \{f,H_T\}_D \ , \qquad (4.4.6)$$

(2) Dirac bracket of any function $f(\underline{X},\underline{p})$ and any second class constraint χ_s vanishes, i.e.,

$$\{f,\chi_s\}_D = 0 \quad (\forall\, \chi_s). \qquad (4.4.7)$$

Thus all second class constraint expressions $\chi_s(\underline{X},\underline{p})$ may be put to zero at will before or after a Dirac bracket calculation. Hence the existence of second class constraints tell us the existence of redundant variables which may be eliminated with the help of $\chi_s(\underline{X},\underline{p})=0$. In other words any system containing redundant variables will have second class constraints.

The rules for quantizing such a system, according to Dirac, are as follows

(1) X^i,p_i go over to operators $\hat{X}^i,\hat{p}_i$ with their commutation relations corresponding to Dirac brackets,

(2) second class constraints may be regarded simply as operator equations

$$\chi_s(\hat{\underline{X}},\hat{\underline{p}}) = 0. \qquad (4.4.8)$$

(3) first class constraints are treated as conditions acting on vectors

$$\eta_m(\hat{\underline{X}},\hat{\underline{p}})\,|\,\rangle = 0 \qquad (4.4.9)$$

and only these vectors $|\,\rangle$ satisfying these conditions may be regarded as physical state vectors.

The above scheme was set up primarily for the quantization of fields, notably the elecromagnetic field in Lorentz guage and Einstein's general relativity. It is pragmatic and mathematically not rigorous.

Let us now return to our original theme to see if Dirac's scheme is applicable to our case of geometric constraints,

Consider the hyperbolic plane V_{-1} embedded as the surface

$$X^3 = \sqrt{1+(X^1)^2+(X^3)^2} \qquad (4.4.10)$$

in M^3. The free quantal motion in V_{-1} is determined by the Hamiltonian

$$H = -(\hbar^2/2m)\nabla^2 \, . \qquad (4.4.11)$$

The operator ∇^2 here is the Laplacian in V_{-1} . If Dirac's scheme does work when the motion is subjected to a geometric constraint, Then it should be possible to view the free motion in V_{-1} as a motion in M^3 described by a suitable Lagrangian, Moreover, when Dirac's scheme is applied, we should be able to recover the geometrical constraint involved and the correct Hamiltonian (4,4,11),

The classical motion in V_{-1} can be described without inconsistency by the Lagrangian

$$\mathcal{L} = \frac{m}{2}\left\{[1+(X^2)^2](\dot{X}^1)^2+[1+(X^1)^2](\dot{X}^2)^2-2X^1X^2\dot{X}^1\dot{X}^2\right\}/[1+(X^1)^2+(X^2)^2]$$

$$-\frac{1}{2}\left(X^3-\sqrt{1+(X^1)^2+(X^2)^2}\right)^2. \qquad (4,4,12)$$

This Lagrangian gives rise to the primary constraint

$$\chi_1 \equiv p_3 = 0, \qquad (4,4,13)$$

Thus the total Hamiltonian is

$$H_T = \frac{m}{2}\left\{[1+(X^1)^2]\,p_1^2 + [1+(X^2)^2]\,p_2^2 + 2X^1X^2p_1p_2\right\} + \frac{1}{2}\left(X^3-\sqrt{1+(X^1)^2+(X^2)^2}\;\right)^2 + u\,p_3\ , \tag{4.4.14}$$

where u is an unknown coefficient. The consistency condition $\dot{p}_3=\{p_3, H_T\}=0$ leads to the secondary constraint

$$\chi_2 = X^3-\sqrt{1+(X^1)^2+(X^2)^2} = 0. \tag{4.4.15}$$

The constraints $\chi_s=0$ (s=1,2) are second class since $\{\chi_1,\chi_2\}= -1$. (u appears in (4.4.14) as a multiplicative factor of the second class constraint $\chi_1=0$, Since χ_1 is to be put to zero, the calculation of u is not necessary). In order to find Dirac bracket

$$\{f_1,f_2\}_D = \{f_1,f_2\}-\{f_1,\chi_s\}\, c_{ss'}\{\chi_{s'},f_2\}\ , \tag{4.5.16}$$

where the matrix ($c_{ss'}$) is defined by $c_{ss'}\{\chi_{s'},\chi_{s''}\}=\delta_{ss''}$, we form first the matrix

$$\begin{pmatrix} \{\chi_1,\chi_1\} & \{\chi_1,\chi_2\} \\ \{\chi_2,\chi_1\} & \{\chi_2,\chi_2\} \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} .$$

The matrix ($c_{ss'}$) is given by

$$(c_{ss'}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

Therefore Dirac bracket has the form

$$\{f_1,f_2\}_D=\{f_1,f_2\}-\{f_1,\chi_1\}\{\chi_2,f_2\}+\{f_1,\chi_2\}\{\chi_1,f_2\}.$$

The Dirac brackets between the variables X^i,p_i are

$$\begin{aligned} &\{X^i,X^j\}_D = \{p_i,p_j\}_D = 0 \quad (i,j=1,2,3) \\ &\{X^i,p_j\}_D = \delta^i_j \quad (i,j=1,2), \\ &\{X^i,p_3\}_D = 0 \quad (i=1,2,3) \end{aligned} \tag{4.4.17}$$

Having determind the second class constraints and found the Dirac brackets between the variables X^i, p_i, we move over to quantum theory arranging these variables into operators satisfying

$$\hat{X}^3 - \sqrt{1+(\hat{X}^1)^2+(\hat{X}^2)^2} = \hat{0}\ , \quad \hat{p}_3 = \hat{0}\ ; \tag{4.4.18a}$$

$$\left\{\begin{array}{l}[\hat{X}^i,\hat{X}^j]_D = [\hat{p}_i,\hat{p}_j]_D = \hat{0} \\ [\hat{X}^i,\hat{p}_j]_D = i\hbar\delta^i_j\end{array}\right. \quad (i,j=1,2)\ . \tag{4.4.18b}$$

Here $[\ ,\ ]_D$ is the commutator corresponding to $\{\ ,\ \}_D$, Now we represent the operators $\hat{X}^i$ $(i=1,2)$ by the multiplication operators X^i, and the operators $\hat{p}_i$ $(i=1,2)$ by the differential operators $-i\hbar\,\partial/\partial X^i$. These operators, of course, act on $L^2(V_{-1})$. To quantize the classical Hamiltonian, we write H in the form $H = \frac{m}{2}(1/\sqrt{g})p_i\sqrt{g}\ g^{ij}p_j$ and then replace X^i, p_i by $\hat{X}^i$ and $\hat{p}_i$. The result is

$$H = -\frac{\hbar^2}{2m}\sqrt{1+(x^1)^2+(x^2)^2}\left\{\frac{\partial}{\partial x^1}\left(\frac{1+(x^1)^2}{\sqrt{1+(x^1)^2+(x^2)^2}}\frac{\partial}{\partial x^1}\right)+\frac{\partial}{\partial x^2}\left(\frac{1+(x^2)^2}{\sqrt{1+(x^1)^2+(x^2)^2}}\frac{\partial}{\partial x^2}\right)\right.$$

$$\left.+\frac{\partial}{\partial x^1}\left(\frac{x^1x^2}{\sqrt{1+(x^1)^2+(x^2)^2}}\frac{\partial}{\partial x^2}\right)+\frac{\partial}{\partial x^2}\left(\frac{x^1x^2}{\sqrt{1+(x^1)^2+(x^2)^2}}\frac{\partial}{\partial x^1}\right)\right\}, \tag{4,4,19}$$

which is identical with (4,4,11).

We note that we could have eliminated X^3 and p_3 by the help of (4,4.13) and (4,4.15) immediately after finding out that χ_s are second class. Then we could have defined new Poisson bracket in terms of the remaining physically significant variables X^i, p_i $(i=1,2)$. In other words, we just despense with the redundant variables X^3, p_3 and work only with the rest of variables. This new Poisson bracket coincides with Dirac bracket for the physically significant variables X^i, p_i $(i=1,2)$.

Dirac's scheme can be applied to any well-behaved hypersurface (not necessarily of constant curvature) in F^N. Also, it is applicable to cases in which more than one geometrical constraint are involved. The applicability of Dirac's scheme to V_{-1} and to the other cases indicated to in this paragraph constitutes a verification of Dirac's "non-rigorous" rules.

GENERAL REMARKS

We have confined our considerations in this thesis to Riemannian manifolds. It is easy, however, to show that most of our results can formally be extended to manifolds with indefinite metrics. Examples of such are the de Sitter spaces. These are 4-dimensional spaces of constant curvature K but of indefinite metric $ds^2 = (dx^i dx^i - dx^4 dx^4)/[1 + \frac{K}{4}(x^i x^i - x^4 x^4)]^2$ $(i = 1,2,3)$ [48]. The infinitesimal generators of the de Sitter groups (the infinitesimal motions of the de Sitter spaces) are easily deduced. These tend to the generators of the Poincare group in the limit $K \to 0$. The formal validity of most of our results for the de Sitter spaces suggests that it might be possible to modify and extend our treatment to tackle the problem of quantization in the de Sitter cosmological models of the universe.

Another problem to pursue is the evaluation of the spectral functions of physical observables in CC^N. These provide the direct link between the theory and experimentally measurable quantities such as expectation values and probability distributions of physical observables.

Finally, we would like to draw the reader's attention to some useful papers which deal with quantization in Riemannian manifolds or with related aspects. These are references [49-57].

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